MA30060: PROBLEM SHEET 1 – SOLUTIONS

1. $$x_{n+1} = f(x_n, \mu) = \mu x_n(1-x_n)$$

a) fixed points

$$x = \mu x(1-x)$$

$\Rightarrow$ $x = 0$ (exists for all μ)

or

$x^* = 1 - \frac{1}{\mu}$ (exists for $\mu \neq 0$)

Non-trivial fixed point is in $[0,1]$ if $\mu > 1$.

b) Period doubling

$$f'(x) = \mu(1-2x).$$

So $f'(0) = \mu$

and $f'(1-\frac{1}{\mu}) = \mu(-1+\frac{2}{\mu}) = 2-\mu.$

Hence $f'(1-\frac{1}{\mu}) = -1$ when $\mu = 3$.

c) 2-cycle

Solve $f^2(x) = x$ to find the 2-cycle.

$$f^2(x) = x \Rightarrow \mu[\mu x(1-x)][1-\mu x(1-x)] = x$$

$$\Rightarrow x(x-1+\tfrac{1}{\mu})(x^2 - x(1+\tfrac{1}{\mu}) + \tfrac{1}{\mu}(1+\tfrac{1}{\mu})) = 0$$

The first two factors are the fixed points of f found in (a).

The quadratic factor gives

$$x_{1,2} = \tfrac{1}{2}(1+\tfrac{1}{\mu}) \pm \tfrac{1}{2}\left[(1+\tfrac{1}{\mu})(1-\tfrac{3}{\mu})\right]^{1/2}$$

$x_{1,2}$ are real if and only if $\mu \geq 3$.

Hence 2-cycle exists for $\mu > 3$, but not $\mu < 3$.
At $\mu = 3$ $\quad x_1 = x_2 = \frac{1}{2}(1+\frac{1}{3}) = \frac{2}{3} = 1 - \frac{1}{3} = 1 - \frac{1}{\mu} = x^*$.

d)

Let $G(x) := f^2(x)$.

Chain rule: $\quad G'(x) = f'(f(x)).f'(x)$.

So, on the 2-cycle $\{x_1, x_2\}$,

$$G'(x_1) = f'(x_2)f'(x_1) = f'(x_1)f'(x_2) = G'(x_2).$$

Hence G' is constant on $\{x_1, x_2\}$.

Stability

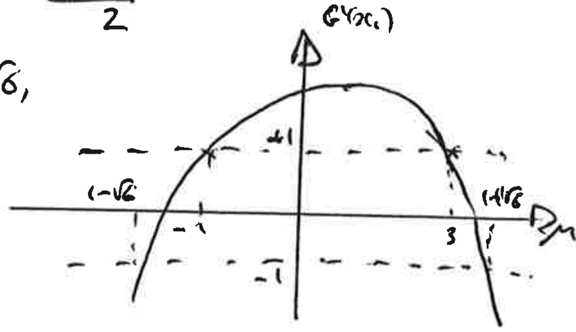
x_1 and x_2 fixed points of G.
Perturbations near 2-cycle decay if $|G'(x)| < 1$.

$$G'(x_1) = \mu^2(1-2x_1)(1-2x_2)$$
$$= 4 + 2\mu - \mu^2.$$

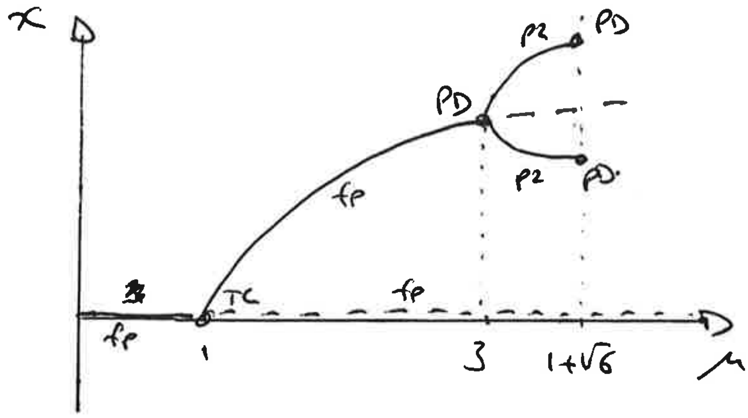
So $\quad G'(x_1) = 1 \quad$ when $\mu^2 - 2\mu - 3 = 0$
$\Rightarrow (\mu-3)(\mu+1) = 0$
$\Rightarrow \mu = -1, 3.$

$G'(x_1) = -1 \quad$ when $\mu^2 - 2\mu - 5 = 0$
$\Rightarrow \mu = \frac{2 \pm \sqrt{4+20}}{2} = 1 \pm \sqrt{6}$

Hence $|G'(x_1)| < 1$ if $3 \leq \mu < 1+\sqrt{6}$, and the 2-cycle is stable in this range.



e) At $\mu = 1+\sqrt{6}$ there is a period-doubling of the 2-cycle. This generates a 4-cycle.



2.

Suppose that the map f has an N-cycle $\{x_0, x_1, \ldots, x_{N-1}\}$.

So $x_{n+1} = f(x_n)$ for $0 \le n \le N-2$
and $x_0 = f(x_{N-1})$

Let $G(x) = f^N(x)$.

Then $G'(x) = f'(f^{N-1}(x)) \frac{d}{dx} f^{N-1}(x)$

$= f'(f^{N-1}(x))\, f'(f^{N-2}(x)) \frac{d}{dx} f^{N-2}(x)$

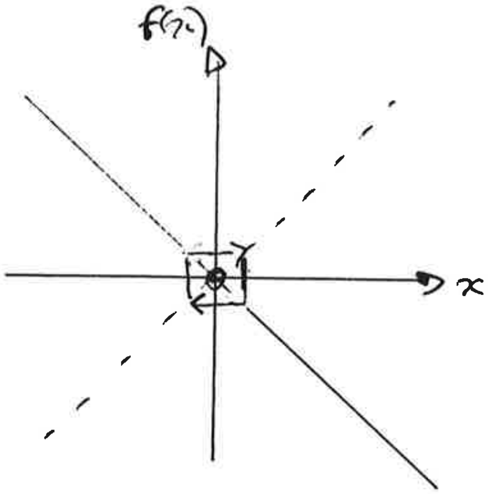
Inductively
$G'(x) = f'(f^{(N-1)}(x))\, f'(f^{N-2}(x)) \ldots f'(f(x))\, f'(x)$

If $x \in \{x_0, x_1, \ldots, x_{N-1}\}$ is a point on an N-cycle then the set $\{f^{N-1}(x), f^{N-2}(x), \ldots, f(x), x\}$ is just a reordering of $\{x_0, x_1, \ldots, x_{N-1}\}$.

Hence $G'(x_j) = \prod_{i=0}^{N-1} f'(x_i)$ for any $j \in \{0, 1, \ldots, N-1\}$

So $G'(x)$ is constant on the N-cycle.

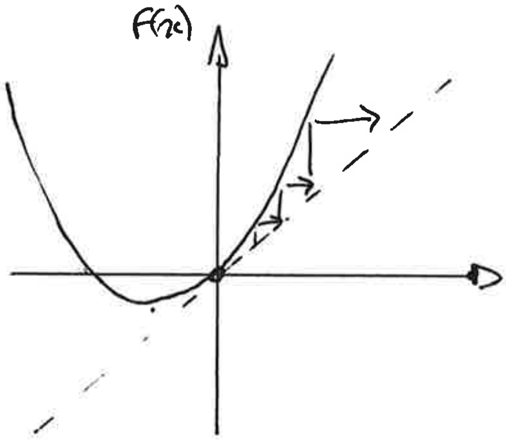
3\. a) $f(x) = -x$



$f^2(x) = x$. So all points except $x=0$ are part of 2-cycles.

$x=0$ is Lyapunov stable but not quasi-asymptotically stable.

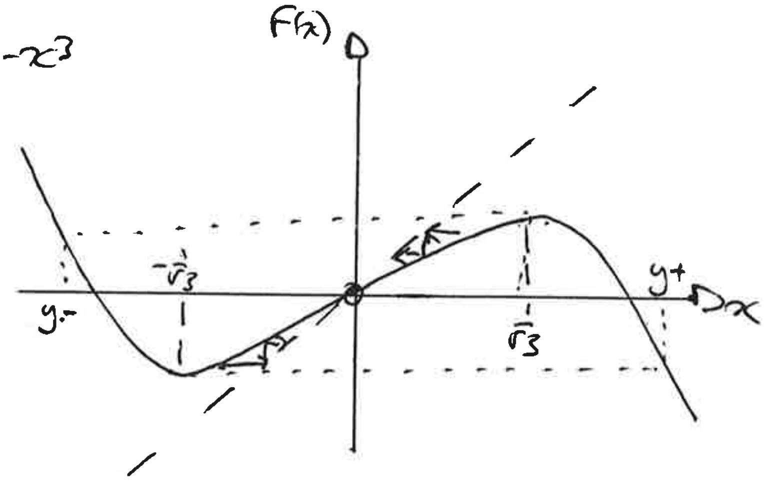
b) $f(x) = x + x^2$



If $x_n > 0$ then $x_n \to \infty$ as $n \to \infty$ since $x_{n+1} - x_n = x_n^2 > 0$

Hence $x=0$ is neither Lyapunov nor ~~a~~ quasi-asymptotically stable.

c) $F(x) = x - x^3$



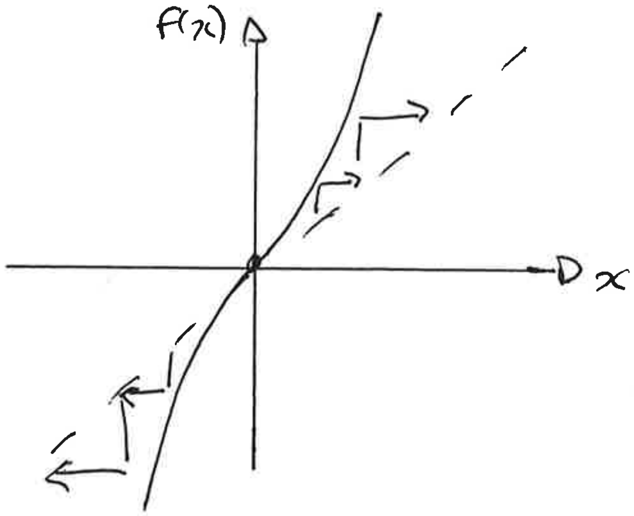
$F(x)$ has turning points at $x = \pm\frac{1}{\sqrt{3}}$.

Let $y^\pm$ be such that $F(y^\pm) = \mp\frac{1}{\sqrt{3}}$.

Then all points in $[y-, y+]$ tend to 0.

Hence $x=0$ is Lyapunov and quasi-asymptotically stable.

d) $f(x) = x + x^3$

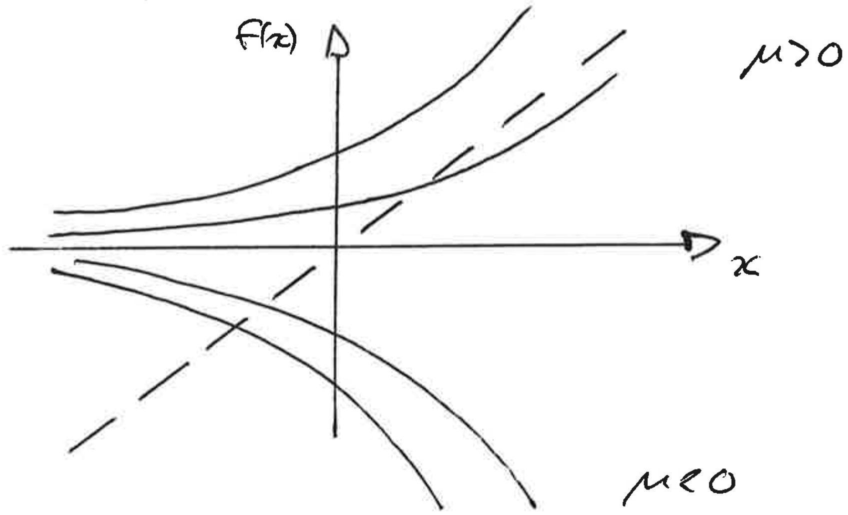


$x=0$ is Lyapunov and quasi-asymptotically unstable.

MA30060 Problem Sheet 2 - Solutions

1.

$$x_{n+1} = \mu e^{x_n} =: f(x_n, \mu)$$



Possibility of saddle-node bifurcation if $\mu > 0$, period doubling if $\mu < 0$.

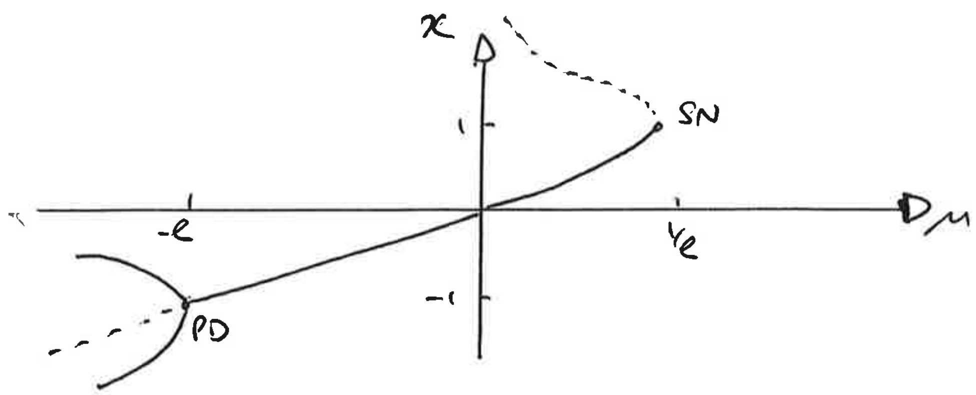
a) Saddle-node bifurcation occurs when $x = f(x,\mu)$ and $f_x(x,\mu) = 1$.

So $x = \mu e^x$ and $\mu e^x = 1$
$\Rightarrow$ $x = 1$ and $\mu = \frac{1}{e}$.

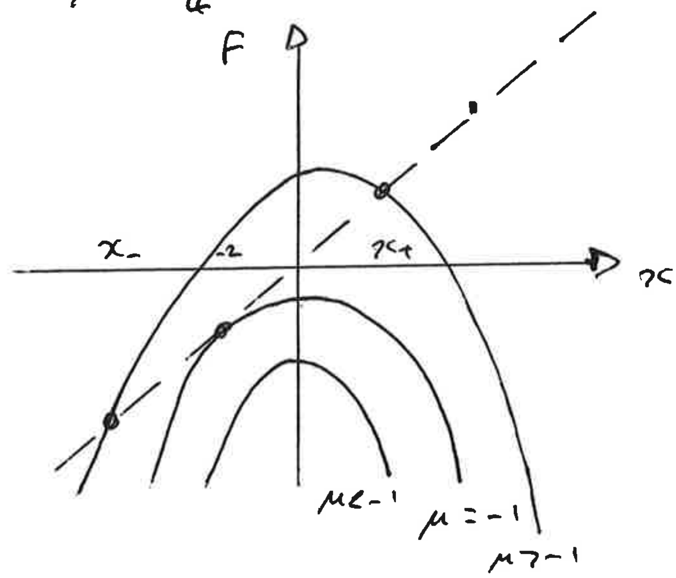
b) Period-doubling bifurcation occurs when $x = f(x,\mu)$ and $f_x(x,\mu) = -1$.

So $x = \mu e^x$ and $\mu e^x = -1$
$\Rightarrow$ $x = -1$ and $\mu = -e$.

Looking at the sketch of $f(x,\mu)$ to infer stability, the bifurcation diagram is



2. $$x_{n+1} = \mu - \frac{x_n^2}{4} =: F(x_n,\mu)$$



Fixed points

$$x = \mu - \frac{x^2}{4}$$

$$\Rightarrow x_\pm = \frac{-1 \pm [1+\mu]^{1/2}}{1/2} = -2(1 \mp [1+\mu]^{1/2}), \quad (\mu \geq -1)$$

Stability

$$F_x(x,\mu) = -\frac{x}{2}.$$

So $F_x(x_+) = 1 - [1+\mu]^{1/2}$

and $|F_x(x_+)| < 1$ if $-1 < \mu < 3$.

Also $F_x(x_-) = 1 + [1+\mu]^{1/2}$

and $|F_x(x_-)| > 1$, for all μ.

Hence: no fixed points ~~as~~ for $\mu < -1$.
at $\mu = -1$, saddle-node bifurcation – pair of stable-unstable fixed points from $x = -2$
at $\mu = 3$, $F_x(x_+, \mu) = -1$, period-doubling bifurcation from $x = 2$

Period-2 points

$$f^2(x) = \mu - \frac{1}{4}\left[\mu - \frac{x^2}{4}\right]^2$$

$$= \mu - \frac{\mu^2}{4} + \frac{\mu x^2}{8} - \frac{x^4}{64}$$

So $x = f^2(x)$

if $x^4 - 8\mu x^2 + 64x + 16x^2 - 64\mu = 0$

$\Rightarrow$ $\underbrace{(x^2 + 4x - 4\mu)}_{\text{fixed points of } F}\underbrace{(x^2 - 4x - 4\mu + 16)}_{\text{Additional fixed points of } f^2} = 0$

$\Rightarrow$ $x_{1,2} = 2 \pm 2[\mu - 3]^{1/2}$ are the period-2 points.

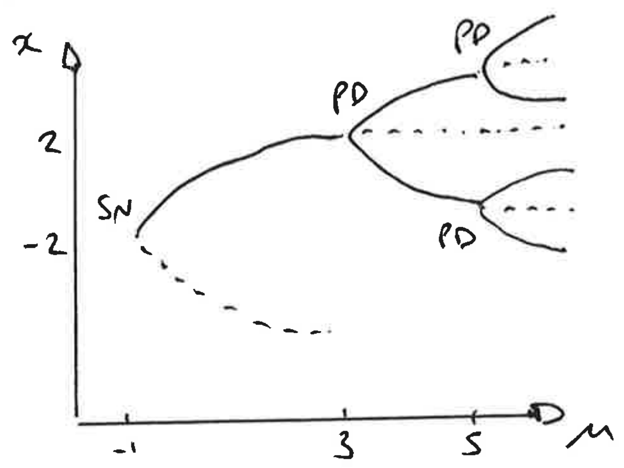
Stability

$$F_x(x_1)F_x(x_2) = \frac{d}{dx} f^2(x)\Big|_{x = x_1 \text{ or } x_2}$$

$$= \frac{1}{4} x_1 x_2$$

$$= 1 - (\mu - 3) = 4 - \mu.$$

Hence period-2 orbit stable for $3 < \mu < 5$, and has a period-doubling bifurcation at $\mu = 5$.



3.

$F: X \to X$, $G: Y \to Y$ conjugate via $h: X \to Y$.

So $h \circ F(x) = G \circ h(x)$.

Let $\{x_0, x_1, \dots, x_{N-1}\}$ be an N-cycle for F.

Set $y_j = h(x_j)$ for $j = 0, 1, \dots, N-1$.

Then
$$\begin{aligned} y_{j+1} &= h(x_{j+1}) \\ &= h \circ F(x_j) = G \circ h(x_j) = G(y_j) \end{aligned}$$
for $j = 0, 1, \dots, N-2$.

Since
$$\begin{aligned} x_0 &= F(x_{N-1}), \\ y_0 &= h(x_0) = h \circ F(x_{N-1}) = G \circ h(x_{N-1}) = G(y_{N-1}). \end{aligned}$$

Hence $\{y_0, y_1 \dots y_{N-1}\}$ is an N-cycle for G.

4.

$$x_{n+1} = f(x_n) = 4x_n(1-x_n) \quad \text{on} \quad X = [0,1]$$

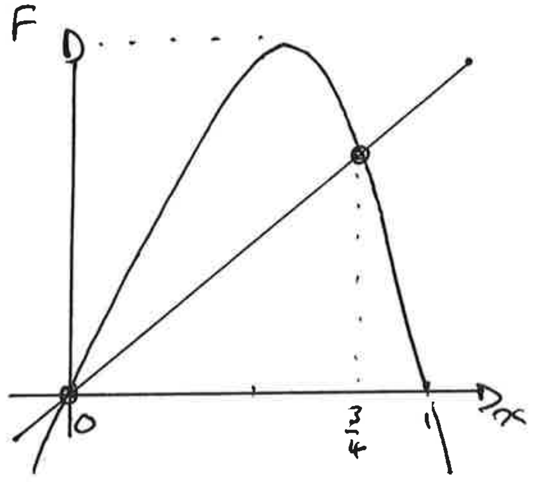
$$y_{n+1} = G(y_n) = 2 - y_n^2 \quad \text{on} \quad Y \subseteq \mathbb{R}$$

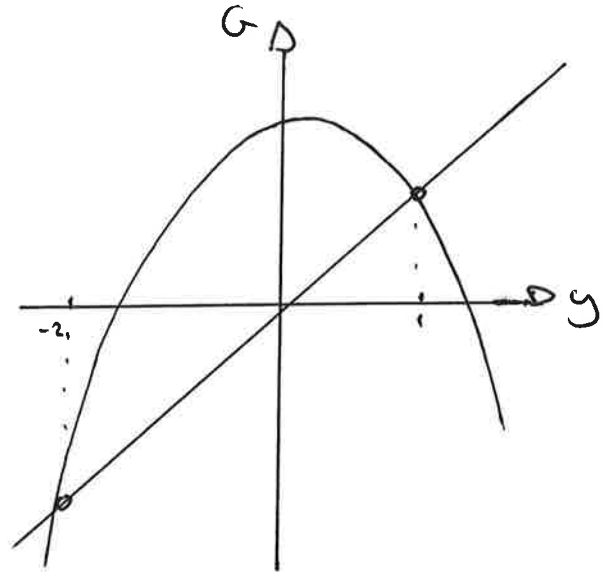
a) F has fixed points when $4x(1-x) = x$
$\Rightarrow 3x - 4x^2 = 0$
$\Rightarrow x = 0$ or $\frac{3}{4}$

G has fixed points when $2-y^2=y$

$\Rightarrow y^2+y-2=0$

$\Rightarrow y=1$ or -2.





Find homeomorphism h
Require $h \circ F(x) = G \circ h(x)$, and fixed points must map to fixed points.

Try $h(x) = ax+b$ for a, b to be found.

If h maps fixed points to fixed points

$$\left.\begin{array}{l} h(0) = -2 \\ h(3/4) = 1 \end{array}\right\} \Rightarrow \begin{array}{l} b=-2 \\ a=4. \end{array}$$

So $h(x) = 4x - 2$ could work, if it is a homeomorphism

Now $h(x)$ is clearly continuous and invertible
$h(X) = h[0,1] = [-2,2] = Y$.

Also
$$
\begin{aligned}
h \circ f(x) &= h(4x(1-x)) \\
&= 16x(1-x) - 2 \\
&= -2 + 16x - 16x^2 \\
&= 2 - (4 - 16x + 16x^2) = 2 - (4x-2)^2 \\
&= 2 - (h(x))^2 \\
&= G \circ h(x)
\end{aligned}
$$

Hence $h(x) = 4x - 2$ provides a conjugacy between all orbits of f and G, not just the fixed points.

b)

$$
\begin{aligned}
f(x, \mu) &= \mu x(1-x) \quad \text{on } X \\
G(y, \lambda) &= \lambda - y^2 \quad \text{on } Y.
\end{aligned}
$$

Let h be a homeomorphism between ~~F~~ and ~~G~~.
Try $h(x) = ax + b$.

Then $h \circ F(x) = G \circ h(x)$ requires

$$
\begin{aligned}
& h(\mu x(1-x)) = \lambda - h(x)^2 \\
\Rightarrow \quad & a\mu x(1-x) + b = \lambda - a^2x^2 - 2abx - b^2 \\
\Rightarrow \quad & a = \mu, \quad b = -\frac{\mu}{2}, \quad \lambda = b^2 + b
\end{aligned}
$$

which means that $\lambda = \frac{\mu}{2}\left(\frac{\mu}{2} - 1\right)$,

or equivalently $\mu = 1 \pm [1 + 4\lambda]^{1/2}$

Hence require $\lambda \geq -1/4$, and to ensure unique μ for any given λ, require $\mu > 1$.

5.

$$F(x) = 2x, \quad G(y) = 3y \quad \text{on} \quad [0, \infty).$$

h is a conjugacy between f and G if

$$G \circ h(x) = h \circ f(x)$$
$$\Rightarrow \quad 3h(x) = h(2x).$$

Hence one requirement is $3h(0) = h(0)$
$\Rightarrow \quad h(0) = 0.$

Try $h(x) = x^{\alpha}$ where α is to be determined.
Then

$$3h(x) = h(2x)$$
$$\Rightarrow \quad 3x^{\alpha} = 2^{\alpha}x^{\alpha}$$
$$\Rightarrow \quad 3 = 2^{\alpha}$$
$$\Rightarrow \quad \alpha = \frac{\log 3}{\log 2}.$$

So $h(x) = x^{\frac{\log 3}{\log 2}}$ works

Clearly $h(x)$ and $h^{-1}(y)$ are continuous on $[0, \infty)$
So h is a homeomorphism as required.

However $h^{-1}(y) = y^{\frac{\log 2}{\log 3}}$ is not differentiable at $y = 0$ since $\frac{\log 2}{\log 3} < 1$.

Hence h is not a diffeomorphism.

MA30060 - PROBLEM SHEET 3 - SOLUTIONS

1. a) $f(x,\mu) = \mu\sin(x)$.

If $x=0$ then $f(x,\mu) = \mu\sin(0) = 0 \quad \forall \mu$.

Hence $x=0$ is a fixed point for all μ.

$$f_x(x,\mu) = \mu\cos(x).$$

At $x=0$, $f_x(0,\mu) = \mu$.

Hence $x=0$ is stable if $-1<\mu<1$ and unstable for μ outside of this range.

Bifurcations occur at $\mu = \pm 1$.

Bifurcation at $\mu = +1$

Let $\hat{\mu} = \mu - 1$.
Then $f(x,\hat{\mu}) = (\hat{\mu}+1)\sin(x)$ has a bifurcation at $x=0, \hat{\mu}=0$.

A Taylor expansion about $x=0$, $\hat{\mu}=0$ gives

$$f(x,\hat{\mu}) = (\hat{\mu}+1)\left(x - \frac{x^3}{3!} + \dots\right)$$
$$= x + \hat{\mu}x - \frac{x^3}{3!} + O(|\hat{\mu}, x|^4)$$

So, with reference to the general classification of bifurcations given in the notes,

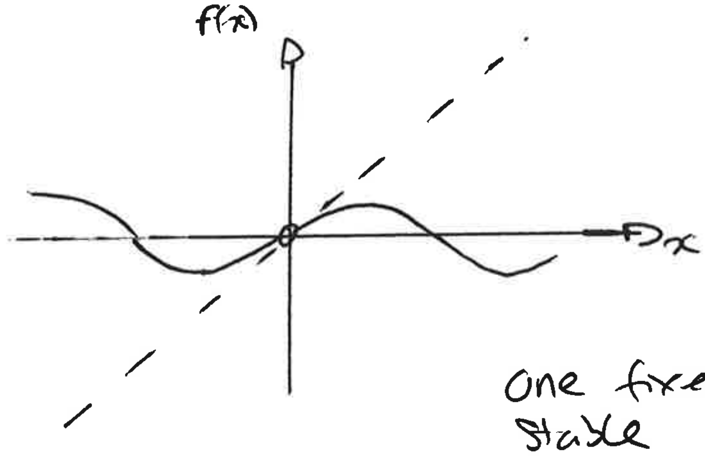
$$a_0 = 1,\ a_1 = 0,\ b_0 = 0,\ b_1 = 1,\ c_0 = -\frac{1}{3!} < 0$$

which indicates a supercritical pitchfork bifurcation.

Sketches

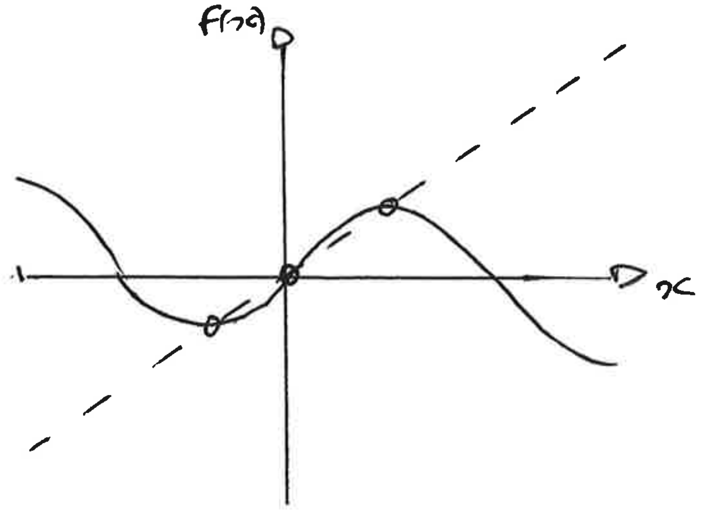
$f(0,\mu)=0 \quad \forall \mu$ and $f_x(0,\mu)=\mu$.

$0<\mu<1$



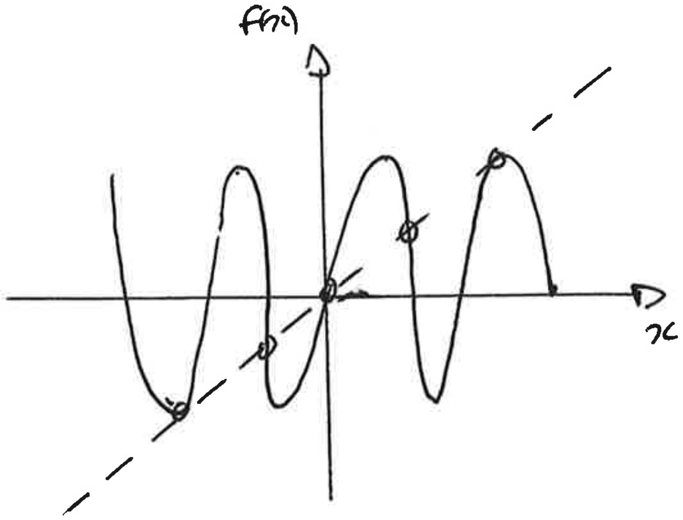
One fixed point, stable

$\mu>1$



Supercritical pitchfork bifurcation creates two new stable fixed points as $x=0$ loses stability.

Then



~~The~~ fixed points from pitchfork have lost stability by period-doubling bifurcation.

Two pairs of new fixed points created by saddle-node bifurcations.

Bifurcation at $\mu = -1$

Let $\hat{\mu} = -(\mu+1)$ (So $\hat{\mu}$ reverses μ axis, and shifts).

Then $f(x,\hat{\mu}) = -(\hat{\mu}+1)\sin(x)$ has a bifurcation at $x=0$, $\hat{\mu}=0$.

A Taylor expansion about $x=0$, $\hat{\mu}=0$ gives

$$F(x,\hat{\mu}) = -(\hat{\mu}+1)\left(x - \frac{x^3}{3!} + \dots\right)$$
$$= -x - \hat{\mu}x + \frac{x^3}{3!} + O(|x,\hat{\mu}|^4).$$

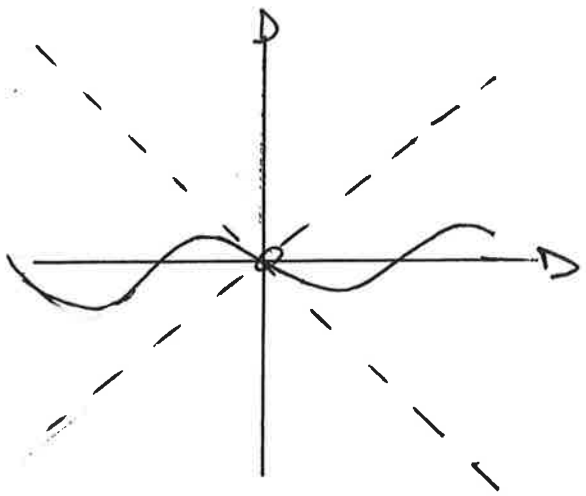
So, with reference to the general classification of bifurcations

$$a_0 = -1,\ a_1 = 0,\ b_0 = 0,\ b_1 = -1,\ c_0 = \frac{1}{3!} > 0$$

which indicates a supercritical period-doubling bifurcation.

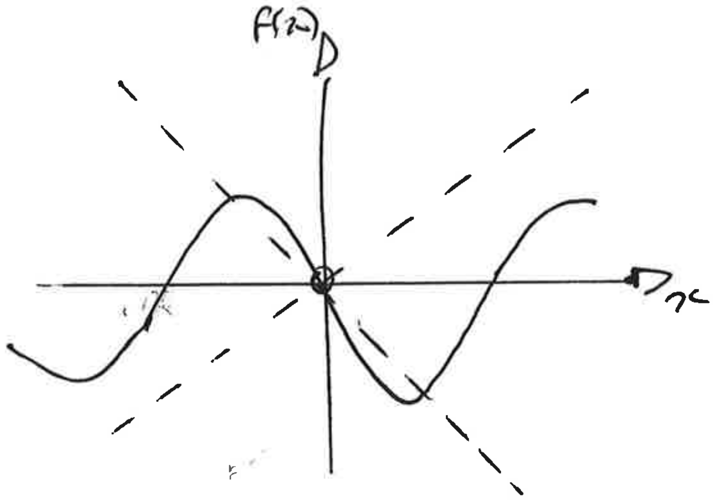
Sketches

$-1 < \mu < 0$



One stable fixed point

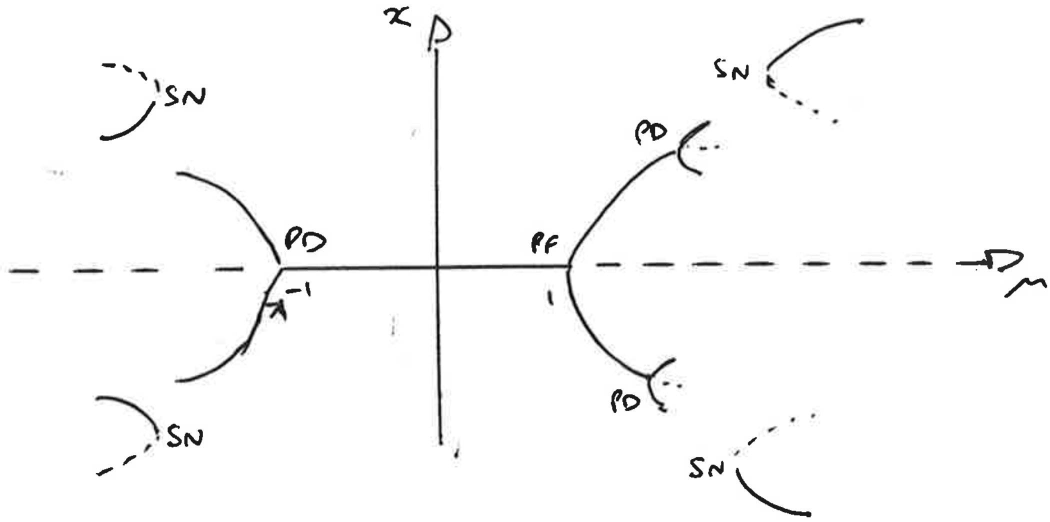
$\mu < -1$



Period-doubling bifurcation with $F_x(0,\mu) = -1$.

Then, as in the case for $\mu > 1$, saddle-node bifurcations create new fixed points whenever $F(x,\mu)$ touches, and then intersects, $y = x$.

Bifurcation diagram



b) $f(x,\mu) = \mu \sinh(x)$

If $x=0$ then $f(x,\mu) = \mu \sinh(0) = 0 \ \forall \mu$.

Hence $x=0$ is a fixed point for all μ.

$f_x(x,\mu) = \mu \cosh(x)$

At $x=0$. $f_x(0,\mu) = \mu$.

Hence $x=0$ is stable if $-1<\mu<1$ and unstable for μ outside this range.

Bifurcations occur at $\mu = \pm 1$.

<u>Bifurcation at $\mu = +1$</u>

Let $\hat{\mu} = \mu - 1$.
Then $f(x,\hat{\mu}) = (\hat{\mu}+1)\sinh(x)$ has a bifurcation at $x=0$, $\hat{\mu}=0$.

A Taylor expansion about $x=0$, $\hat{\mu}=0$ gives

$$f(x,\hat{\mu}) = (\hat{\mu}+1)\left(x + \frac{x^3}{3!} + \dots\right)$$
$$= x + \hat{\mu}x + \frac{x^3}{3!} + O(|\hat{\mu},x|^4)$$

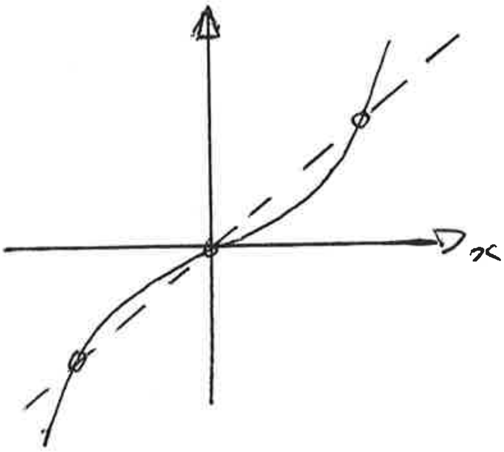
With reference to the general classification

$$a_0 = 1, \ a_1 = 0, \ b_0 = 0, \ b_1 = 1, \ c_0 = \frac{1}{3!} > 0$$

which indicates a subcritical pitchfork bifurcation.

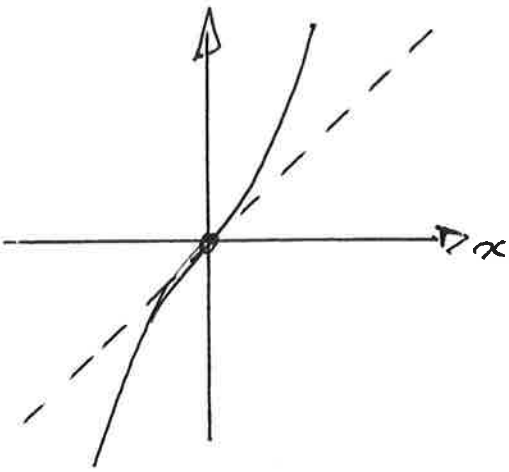
Sketches

$0 < \mu < 1$



Two non-zero fixed points, both unstable. One stable fixed point, at $x=0$.

$\mu > 1$



Single fixed point, unstable, at $x=0$, after unstable fixed point pair contracted with increasing μ, vanishing when $\mu = 1$.

Bifurcation at $\mu = -1$

Let $\hat{\mu} = -(\mu+1)$

Then $F(x,\hat{\mu}) = -(\hat{\mu}+1)\sinh(x)$ has a bifurcation at $x=0$, $\hat{\mu}=0$.

Taylor expansion about $x=0$, $\hat{\mu}=0$ gives

$$F(x,\hat{\mu}) = -(\hat{\mu}+1)\left(x + \frac{x^3}{3!} + \dots\right)$$

$$= -x - \hat{\mu}x - \frac{x^3}{3!} + O(|\hat{\mu},x|^4)$$

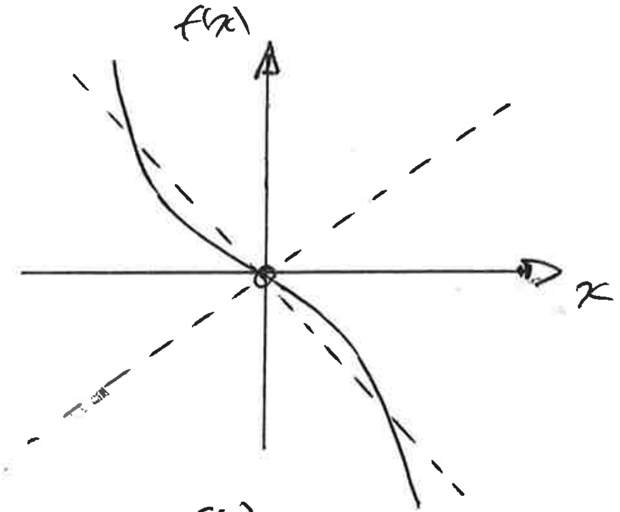
So, with reference to the general classification of bifurcations

$$a_0 = -1,\ a_1 = 0,\ b_0 = 0,\ b_1 = -1,\ c_0 = -\frac{1}{3!}\ \text{etc.}$$

which indicates a subcritical period-doubling bifurcation.

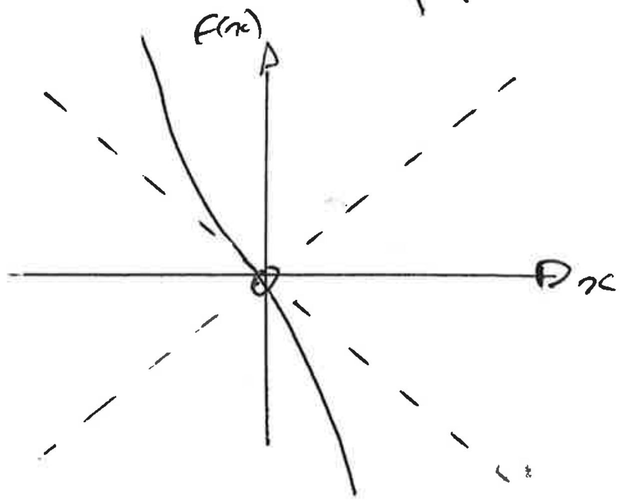
Sketches

$-1 < \mu < 0$



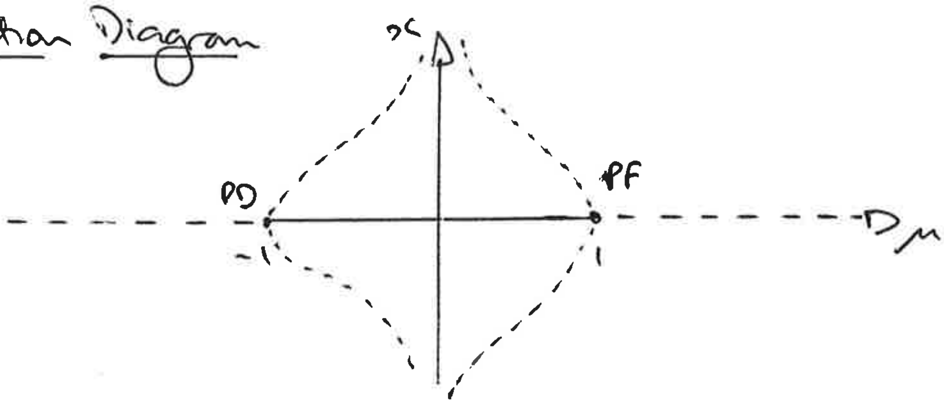
stable fixed point at $x = 0$, unstable 2-cycle

$\mu < -1$



unstable fixed point at $x = 0$ after 2-cycle has contracted as μ decreases, vanishing when $\mu = -1$.

Bifurcation Diagram



2. Sawtooth map $x_{n+1} = 2x_n \bmod 1$.

a) Let $x_0 = \frac{1}{7}$.

Then $x_1 = \frac{2}{7}$, $x_2 = \frac{4}{7}$, $x_3 = \frac{8}{7} = \frac{1}{7} = x_0$.

Hence $\frac{1}{7}$ is part of the 3-cycle $\{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\} = C_1$.

Let $x_0 = \frac{3}{7}$.

Then $x_1 = \frac{6}{7}$, $x_2 = \frac{12}{7} \to \frac{5}{7}$, $x_3 = \frac{10}{7} \to \frac{3}{7} = x_0$.

Hence $\frac{3}{7}$ is part of the 3-cycle $\{\frac{3}{7}, \frac{6}{7}, \frac{5}{7}\} = C_2$.

~~By~~ The action of F corresponds to shifting the binary sequence for x one place left. So points in 3-cycles have period-3 binary expansions.

Try $0.001\,001\ldots = \frac{1}{8}\left(1 + \frac{1}{8} + \frac{1}{8^2} + \ldots\right)$
$\quad\frac{1}{2}\,\frac{1}{4}\,\frac{1}{8}$

$= \frac{1}{8}\left(\frac{1}{1-\frac{1}{8}}\right) = \frac{1}{7}$.

Try $0.011\,011\ldots = \frac{1}{4} + \frac{1}{8} + \frac{1}{32} + \frac{1}{64} + \ldots$ etc

or, observe $0.011\,011\ldots = \frac{1}{7} + \frac{2}{7} = \frac{3}{7}$.

b) Let $x_0 = \frac{p}{7.2^k}$ for integers $1 \le p \le 6$, $k \geq 0$.

Then $x_1 = \frac{p}{7.2^{k-1}}$, $x_2 = \frac{p}{7.2^{k-2}}, \ldots, x_k = \frac{p}{7} < 1$

Then, if $p \in \{1, 2, 4\}$, $x_k \in \{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$ and the orbit is on the 3-cycle C_1.

But if $p \in \{3, 5, 6\}$, $x_k \in \{\frac{3}{7}, \frac{5}{7}, \frac{6}{7}\}$ and the orbit is on the 3-cycle C_2.

Hence
$$O^+(x_0) = \begin{cases} \left\{ \frac{p}{7.2^k}, \frac{p}{7.2^{k-1}}, \cdots \frac{p}{7.2} \right\} \cup C_1 & \text{if } p \in \{1, 2, 4\} \\ \left\{ \frac{p}{7.2^k}, \frac{p}{7.2^{k-1}}, \cdots \frac{p}{7.2} \right\} \cup C_2 & \text{if } p \in \{3, 5, 6\} \end{cases}$$

3. Sawtooth map $x_{n+1} = 2x_n \bmod 1$.

a) Binary expansions for initial points on 4-cycles include

$$x_0 = 0.0001\ 0001 \ldots$$
$$x_0 = 0.0011\ 0011 \ldots$$
$$x_0 = 0.0111\ 0111 \ldots$$

The point $x_0 = 0.01010101\ldots$ is on a 2-cycle. All other symbol sequences are shifted copies of those above.

The three initial points given above correspond to distinct 4-cycles because each symbol sequence contains blocks of 1s of different length.

The 4-cycles are:

$$x_0 = 0.0001\ldots = \frac{1}{16}\left(\frac{1}{1-1/16}\right) = \frac{1}{15} \to \frac{2}{15} \to \frac{4}{15} \to \frac{8}{15} \to \frac{1}{15}$$

$$x_0 = 0.0011\ldots = \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5} \to \frac{2}{5} \to \frac{4}{5} \to \frac{3}{5} \to \frac{1}{5}.$$

$$x_0 = 0.0111\ldots = \frac{1}{15} + \frac{2}{15} + \frac{4}{15} = \frac{7}{15} \to \frac{14}{15} \to \frac{13}{15} \to \frac{11}{15} \to \frac{7}{15}.$$

b) Binary expansions for initial points on 5-cycles are generated in a similar way:

$$x_0 = 0.00001\ 00001\ldots = \frac{1}{31}$$
$$x_0 = 0.00011\ 00011\ldots = 3/31$$
$$x_0 = 0.00111\ 00111\ldots = 7/31$$
$$x_0 = 0.01111\ 01111\ldots = 15/31$$
$$x_0 = 0.00101\ 00101\ldots = 5/31$$
$$x_0 = 0.01011\ 0.01011\ldots = 11/31.$$

~~The~~ Any other symbol sequence of least period 5 is a shifted copy of one of those above.

Then

$$x_0 = \frac{1}{31} \to \frac{2}{31} \to \frac{4}{31} \to \frac{8}{31} \to \frac{16}{31} \to \frac{1}{31}$$

$$x_0 = \frac{3}{31} \to \frac{6}{31} \to \frac{12}{31} \to \frac{24}{31} \to \frac{17}{31} \to \frac{3}{31}$$

$$x_0 = \frac{7}{31} \to \frac{14}{31} \to \frac{28}{31} \to \frac{25}{31} \to \frac{19}{31} \to \frac{7}{31}$$

$$x_0 = \frac{15}{31} \to \frac{30}{31} \to \frac{29}{31} \to \frac{27}{31} \to \frac{23}{31} \to \frac{15}{31}$$

$$x_0 = \frac{5}{31} \to \frac{10}{31} \to \frac{20}{31} \to \frac{9}{31} \to \frac{18}{31} \to \frac{5}{31}$$

$$x_0 = \frac{11}{31} \to \frac{22}{31} \to \frac{13}{31} \to \frac{26}{31} \to \frac{21}{31} \to \frac{11}{31}.$$

MA30060 - PROBLEM SHEET 4 - SOLUTIONS

1. a) $$x_{n+1} = 4\mu - (\mu+3)x_n + x_n^2$$

Fixed points

$$x = 4\mu - (\mu+3)x + x^2$$
$$\Rightarrow 0 = x^2 - (\mu+4)x + 4\mu$$
$$\Rightarrow 0 = (x-\mu)(x-4)$$
$$\Rightarrow x = 4 \text{ or } x = \mu.$$

Both of these fixed points exist for all μ.

Stability

$$f_x(x,\mu) = -(\mu+3) + 2x$$

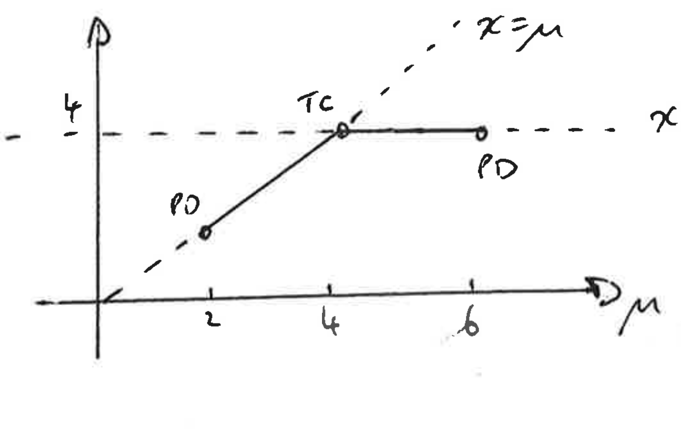
So $f_x(4,\mu) = 5-\mu$ and $x=4$ is stable for $4<\mu<6$.

$f_x(\mu,\mu) = \mu - 3$ and $x=\mu$ is stable for $2<\mu<4$.

At $\mu=2$, $f_x(\mu,\mu) = -1$, so looks like period-doubling ~~fixed~~ bifurcation at fixed point $x=\mu$.

At $\mu=4$, $f_x(4,\mu) = +1$ and $f_x(\mu,\mu) = +1$. Both fixed points exist either side of $\mu=4$, so looks like transcritical bifurcation.

At $\mu=6$, $f_x(4,\mu) = -1$, looks like period-doubling bifurcation at fixed point $x=4$.



Near $\mu = 4$

Let $y = x - 4$, $\lambda = \mu - 4$.

Write $x_{n+1} = 4\mu - (\mu+3)x_n + x_n^2$

as $x_{n+1} = (4 - x_n)(\mu - x_n) + x_n$

Then
$$
\begin{aligned}
y_{n+1} &= x_{n+1} - 4 \\
&= -y_n(\lambda - y_n) - 4 + x_n \\
&= -\lambda y_n + y_n^2 + y_n \\
&= (1-\lambda)y_n + y_n^2. \\
&= y_n - \lambda y_n + y_n^2
\end{aligned}
$$

So, with refeence to the bifurcation classification table, $a_0 = 1$, $a_1 = 0$, $b_0 = 1$, $b_1 = -1$ which indicates a transcritical bifurcation.

Near $\mu = 2$

Let $y = x - 2 - \lambda$, $\lambda = \mu - 2$.

So $\mu - x = -y$ and $4 - x = 2 - y - \lambda$.

Then
$$
\begin{aligned}
y_{n+1} &= x_{n+1} - 2 - \lambda \\
&= (4 - x_n)(\mu - x_n) + x_n - 2 - \lambda \\
&= (2 - \lambda - y_n)(-y_n) + y_n \\
&= (\lambda - 1)y_n + y_n^2 \\
&= -y_n + \lambda y_n + y_n^2
\end{aligned}
$$

So $a_0 = -1$, $a_1 = 0$, $b_0 = 1$, $b_1 = 1$, which indicates a period-doubling bifurcation,

Consider second iterate to determine criticality:

$$
\begin{aligned}
y_{n+2} &= \cancel{(-1\lambda)}\,(\lambda - 1)[(\lambda-1)y_n + y_n^2] + [(\lambda-1)y_n + y_n^2]^2 \\
&= (\lambda^2 - 2\lambda + 1)y_n + (\lambda - 1)y_n^2 + (\lambda^2 - 2\lambda + 1)y_n^2 \\
&\quad + y_n^4 + 2(\lambda - 1)y_n^3
\end{aligned}
$$

$$= (1-2\lambda)y_n + (\lambda^2-\lambda)y_n^2 - 2y_n^3 + \lambda^2 y_n + O(|\lambda, y_n|^4)$$
$$= (1-2\lambda)y_n - 2y_n^3 - \lambda y_n^2 + \lambda^2 y_n + O(|\lambda, y_n|^4).$$

So, 2-cycle exists ($y_{n+2} = y_n$) if $y \sim O(\lambda^{1/2})$ and $y \approx (1-2\lambda)y - 2y^3$.

$\Rightarrow \lambda \approx -y^2 < 0$

$\Rightarrow \mu < 2.$

If $\mu < 2$, the fixed point at $x = \mu$ is unstable. Hence the 2-cycle is stable and the period-doubling bifurcation is supercritical.

<u>Near $\mu = 6$</u>

Let $y = x - 4$, $\lambda = \mu - 6$

Then
$$y_{n+1} = x_{n+1} - 4$$
$$= (4 - x_n)(\mu - x_n) + x_n - 4$$
$$= (-y_n)(\lambda + 2 - y_n) + y_n$$
$$= -(1+\lambda)y_n + y_n^2.$$
$$= -y_n - \lambda y_n + y_n^2$$

So $a_0 = -1$, $a_1 = 0$, $b_0 = -1$, $b_1 = 1$ which indicates a period doubling bifurcation.

The second iterate is

$$y_{n+2} = (1+2\lambda)y_n - 2y_n^3 + \lambda y_n^2 + \lambda^2 y_n + O(|y, \lambda|^4)$$

So 2-cycle exists if $y \sim O(\lambda^{1/2})$ and $y \approx (1+2\lambda)y - 2y^3 \Rightarrow \lambda \approx y^2 > 0.$

$\Rightarrow \mu > 6.$

If $\mu > 6$ then fixed point $x = 4$ unstable. So period-doubling bifurcation supercritical.

b) $$x_{n+1} = \mu - 2 + \mu x_n + x_n + 3x_n^2 + x_n^3 =: f(x_n, \mu).$$

<u>fixed points</u>

$$x = f(x, \mu)$$
$$\Rightarrow \quad 0 = \mu - 2 + \mu x + 3x^2 + x^3$$
$$= (x+1)(x^2 + 2x + \mu - 2)$$
$$\Rightarrow \quad x = -1 \quad \text{or} \quad x_\pm = -1 \pm \sqrt{3-\mu} \quad (\text{exists if } \mu < 3).$$

<u>Stability</u>

$$f_x(x, \mu) = \mu + 1 + 6x + 3x^2$$

So $f_x(-1, \mu) = \mu - 2$, stable if $1 < \mu < 3$.

$$f_x(x_+, \mu) = \mu + 1 + 6\left[-1 + \sqrt{3-\mu}\right] + 3\left[-1 + \sqrt{3-\mu}\right]^2$$
$$= \mu + 1 - 6 + 6\sqrt{3-\mu} + 3\left[1 - 2\sqrt{3-\mu} + (3-\mu)\right]$$
$$= \mu - 2 + 9 - 3\mu$$
$$= 7 - 2\mu.$$

Similarly $f_x(x_-, \mu) = 7 - 2\mu$.

So $x_\pm$ unstable if they exist ($\mu < 3$).

<u>Bifurcation</u>

At $\mu = 3$, three fixed points become one (as μ increases). For $\mu < 3$ two of these fixed points are unstable.

So subcritical pitchfork bifurcation at $\mu = 3$.

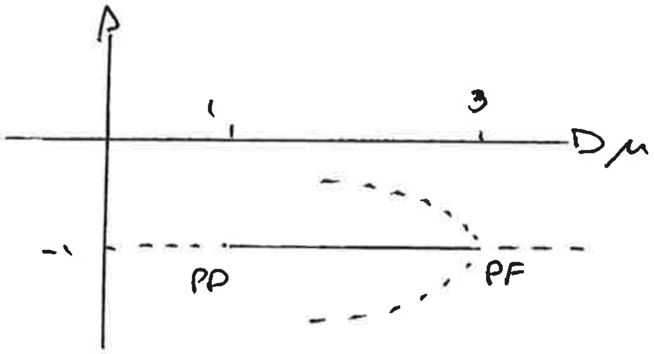
<u>Canonical form</u>

Let $y = x + 1$ and $t = \mu - 3$.

Then
$$y_{n+1} = x_{n+1} + 1$$
$$= (x_n + 1)(x_n^2 + 2x_n + \mu - 2) + x_n + 1$$

$$= y_n(y_n^2 + \lambda) + y_n$$
$$= (1+\lambda)y_n + y_n^3$$

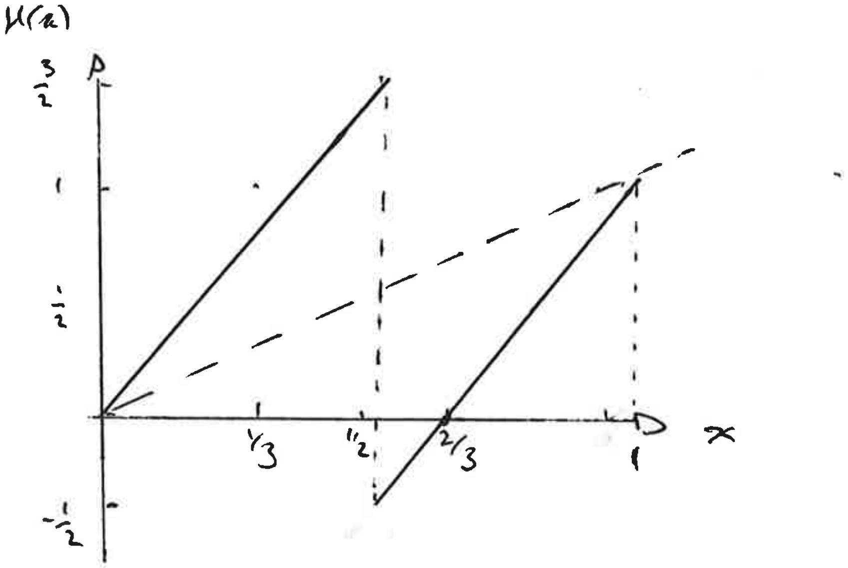
This is the canonical form for a subcritical pitchfork bifurcation.



2.

$$H(x) = \begin{cases} 3x & 0 \le x \le 1/2 \\ 3x-2 & 1/2 < x \le 1. \end{cases}$$

i)

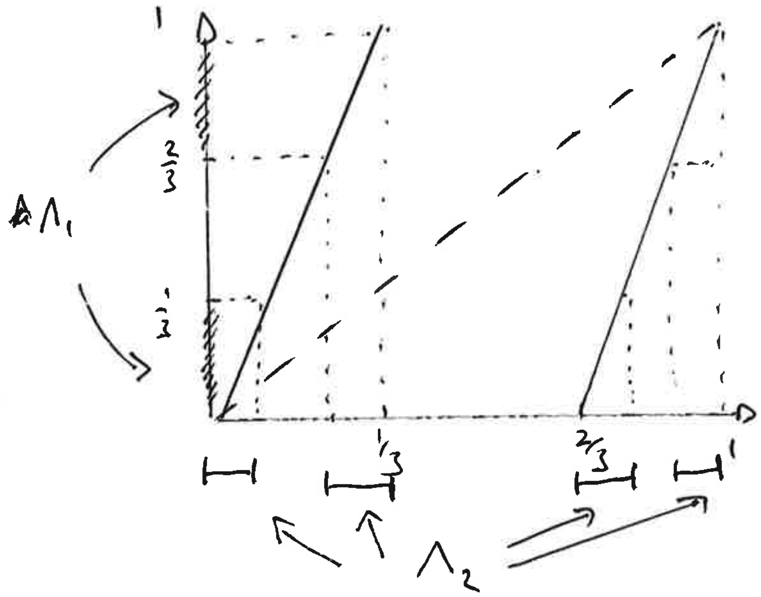


Points that remain in $[0,1]$ for at least one iteration

$$\Lambda_1 = [0,1/3] \cup [2/3,1]$$

Points that remain in $[0,1]$ for at least two iterations

$$\Lambda_2 = H^{-1}(\Lambda_1)$$
$$= [0, 1/9] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$



So Λ_2 is a disjoint union of 4 closed intervals.

Inductively, Λ_n is a disjoint union of 2^n closed intervals.

These intervals are nested so that $\Lambda_n \supset \Lambda_{n+1}$.

Hence $\Lambda := \bigcap_{k=1}^{\infty} \Lambda_n$ is non-empty by the Cantor Intersection Theorem.

ii) ~~An~~ Let x have the base-3 expansion

$$x = 0.\overset{\frac{1}{3}}{a_0}\,\overset{\frac{1}{9}}{a_1}\,\overset{\frac{1}{27}}{a_2}\ldots$$

The action of H ~~transformation~~ on x is conjugate to the action of the shift

σ on the base-3 expansion, possibly with some adjustment of the leading term resulting from the -2 if $x > 1/2$.

A single iteration of ~~f~~ ~~H~~ maps x outside $[0,1]$ if $1/3 < x < 2/3$.

In base-3 terms, this is if x is between $0.100\ldots$ and $0.111\ldots$

All other base-3 expansions, including $0.100\ldots$ and $0.111\ldots$, map inside $[0,1]$.

Hence any ~~base~~ point x ~~contain~~ with a base-3 expansion containing a 1 will eventually be shifted under σ to an expansion $0.1a_1a_2a_3\ldots$ and will map outside $[0,1]$ on next iteration.

The points $x = \frac{1}{3} = 0.100\ldots$ and $x = \frac{2}{3} = 0.111\ldots$ correspond to the pre-images of the fixed point $x=1$.

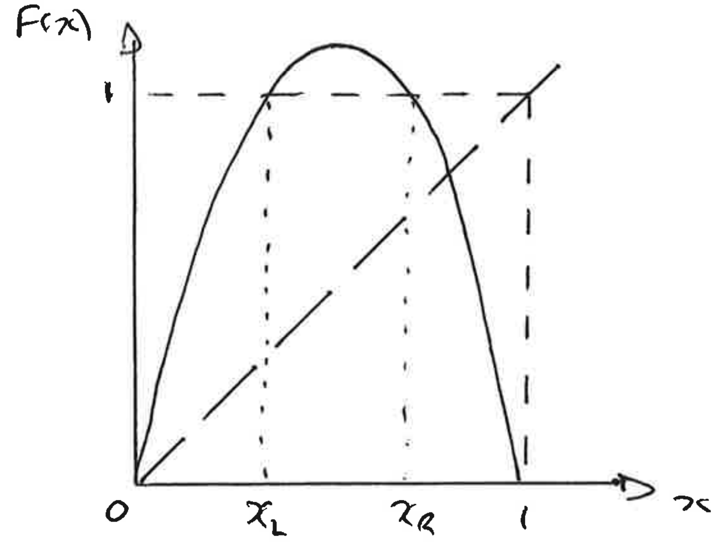
In summary $\Lambda = \{0.a_0a_1a_2\ldots \mid a_i \neq 1 \;\forall i\} \cup \{0.1000\ldots, 0.111\ldots\}$

iii) $$0.002\,002\,002\ldots = \frac{2}{27}\left(1 + \frac{1}{27} + \left(\frac{1}{27}\right)^2 + \ldots\right)$$
$$= \frac{2}{27}\left(\frac{1}{1-1/27}\right) = \frac{1}{13}.$$

Iterating $H(1/13)$:
$$\frac{1}{13} \to \frac{3}{13} \to \frac{9}{13} \to \frac{1}{13},$$ period 3, as expected.

3. $I = [0,1]$. $f(x) = \mu x(1-x)$. $\mu > 2+\sqrt{5}$.

Find $f^{-1}(I)$



$F(x)=1$ if $\mu x(1-x) = 1$

$\Rightarrow \quad x - x^2 = \frac{1}{\mu}$

$\Rightarrow \quad x = \dfrac{1 \pm [1-\frac{4}{\mu}]^{1/2}}{2} = \underset{-}{x_L}, \underset{+}{x_R}$

Note x_L, x_R exist, and are distinct, if $\mu > 4$.

Evaluate gradient on $I \cap f^{-1}(I) = [0, x_L] \cup [x_R, 1]$

$f_x(x) = \mu(1-2x)$

On $[0, x_L]$, $f_x(x)$ takes its minimum value (which is positive) at $x = x_L$:

$$f_x(x_L) = \mu\left(1 - \left(1 - \sqrt{1-\tfrac{4}{\mu}}\right)\right) = \mu\sqrt{1-\tfrac{4}{\mu}}.$$

On $[x_R, 1]$, $f_x(x)$ takes maximum (i.e. least negative) value at $x = x_R$:

$$f_x(x_R) = -\mu\sqrt{1-\frac{4}{\mu}}.$$

Hence $|f_x(x_R)| = |f_x(x_L)| > 1$
when $\sqrt{\mu^2-4\mu} > 1$
$\Rightarrow \mu^2 - 4\mu - 1 > 0$

Equality occurs when $\mu = \dfrac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$.

Since x_L, x_R require $\mu > 4$ for existence,
$|f_x(x_R)| = |f_x(x_L)| > 1$ if $\mu > 2+\sqrt{5}$.

Then, for $x \in [0, x_L] \cup [x_R, 1] = I \cap f^{-1}(2)$,

$$|f_x(x)| > |f_x(x_L)| > \lambda > 1.$$

4.

$F: \mathbb{R}_+ \to \mathbb{R}_+$, $\quad F(x) = 2x$
$G: \mathbb{R} \to \mathbb{R}$, $\quad G(y) = y + \log(2)$.

<u>F and G conjugate</u>

Let $h(x) = \log(x)$.

Then
$$\begin{aligned} h \circ F(x) &= \log(2x) \\ &= \log(2) + \log(x) \\ &= \log(2) + h(x) = G(h(x)) \\ &\qquad\qquad\qquad\;\; = G \circ h(x). \end{aligned}$$

Hence F and G are conjugate via h (which is continuous on $\mathbb{R}^+$ and has a continuous inverse on $\mathbb{R}$).

SDIC

F has SDIC:

fix $\delta = 1$, say. for any $x > 0$ and $\varepsilon > 0$ choose $y > 0$ and $n \geq 1$ such that

$$\frac{\varepsilon}{2} < |x - y| < \varepsilon$$

and

$$\frac{1}{2^{n-1}} < \varepsilon.$$

Then $|F(x) - F(y)| = 2|x - y|$

$$\Rightarrow |F^n(x) - F^n(y)| > 2^n \frac{\varepsilon}{2} > 1 = \delta.$$

But $|x - y| < \varepsilon$.

Hence F has SDIC.

G does not have SDIC:

for any $x, y \in \mathbb{R}$, $G(x) - G(y) = x - y$
$\Rightarrow G^n(x) - G^n(y) = x - y \quad \forall n \geq 1.$

Hence G does not have SDIC.

Hence SDIC is not preserved under conjugacy.

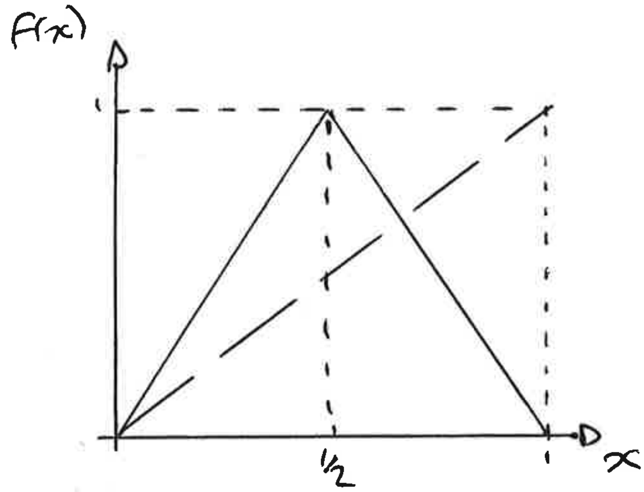
MA30060 - PROBLEM SHEET 5 - SOLUTIONS

1. Tent map

$$x_{n+1} = f(x_n)$$

where
$$f(x) = \begin{cases} 2x, & 0 \le x \le \tfrac{1}{2} \\ 2(1-x), & \tfrac{1}{2} \le x \le 1 \end{cases}$$

i)



ii) Let the point $x \in [0,1]$ have binary expansion $x = 0.a_0 a_1 a_2 \ldots$

$0 \le x < \tfrac{1}{2}$ if and only if $a_0 = 0$.

Then $f(x) = 2x$, corresponds to shifting the binary expansion one place to the left: $f(x) = 0.a_1 a_2 a_3 \ldots$

$\tfrac{1}{2} \le x \le 1$ if and only if $a_1 = 0$.

Then $1-x$ corresponds to $0.1111\ldots - 0.a_0 a_1 a_2 \ldots = 0.0\bar{a}_1 \bar{a}_2 \ldots$

where $\bar{a}_i = \begin{cases} 1 & \text{if } a_i = 0 \\ 0 & \text{if } a_i = 1. \end{cases}$

So $f(x) = 2(1-x)$ corresponds to

$f(x) = 0.\bar{a}_1 \bar{a}_2 \bar{a}_3 \ldots$

Hence the action of F is equivalent to the action of $\hat{\sigma}$ where

$$\hat{\sigma}(0.a_0a_1\ldots) = \begin{cases} 0.a_1a_2\ldots & , a_0=0 \\ 0.\bar{a}_1\bar{a}_2\ldots & , a_0=1 \end{cases}$$

and $\bar{a}_i = 1-a_i$.

iii) Points in 3-cycles under f correspond to symbol sequences that have period 3 under $\hat{\sigma}$.

Any such symbol sequence must contain at least one 0, and at least one 1. Hence it must contain the block 01.

Without loss of generality put this block at the start of the sequence.

So, let $x_0 = 0.01a_2a_3a_4a_5\ldots$

Then
$$x_1 = 0.1a_2a_3a_4\ldots$$
$$x_2 = 0.\bar{a}_2\bar{a}_3\bar{a}_4\ldots$$

There are two possibilities for x_3.
If $\bar{a}_2=0$, $a_2=1$ and

$$x_3 = 0.\bar{a}_3\bar{a}_4\bar{a}_5\ldots$$

Hence $x_3=x_0$ if
$$\bar{a}_3 = a_0 \Rightarrow a_3=1$$
$$\bar{a}_4 = a_1 \Rightarrow a_4=0$$
$$\bar{a}_5 = a_2 \Rightarrow a_5=0$$
$$\bar{a}_6 = a_3 \Rightarrow a_6=0 \quad \text{etc}$$

Hence $x_0 = 0.0111000111\ldots = \frac{64}{63}\left(\frac{7}{16}\right)=\frac{4}{9}.$

if $\bar{a}_2 = 1$, $a_2 = 0$ and

$$x_3 = 0\cdot a_3 a_4 a_5 \ldots$$

Hence $x_3 = x_0$ if

$$\begin{aligned} a_3 &= a_0 = 0 \\ a_4 &= a_1 = 1 \\ a_5 &= a_2 = 0 \quad \text{etc} \end{aligned}$$

Hence $x_0 = 0\cdot 010010 \ldots = \frac{8}{7}\cdot\frac{2}{8} = \frac{2}{7}.$

The complete 3-cycles are:

$$\frac{4}{9} \to \frac{8}{9} \to \frac{2}{9} \to \frac{4}{9} \ldots$$

$$\frac{2}{7} \to \frac{4}{7} \to \frac{6}{7} \to \frac{2}{7} \ldots$$

2. Let $f: I \to \mathbb{R}$ have a 4-cycle $\{x_0, x_1, x_2, x_3\}$.

i) a) Suppose $x_0 < x_1 < x_2 < x_3$.
Let $I_0 = [x_0, x_1]$, $I_1 = [x_1, x_2]$, $I_2 = [x_2, x_3]$.

Then
$$\begin{aligned} f(I_0) &\supseteq [f(x_0), f(x_1)] = [x_1, x_2] = I_1 \\ f(I_1) &\supseteq [f(x_1), f(x_2)] = [x_2, x_3] = I_2 \\ f(I_2) &\supseteq [f(x_3), f(x_2)] = [x_0, x_3] = I_0 \cup I_1 \cup I_2. \end{aligned}$$

Hence the transition graph Γ of f-covering relations is

$$\begin{tikzcd} I_0 \arrow[rr] & & I_1 \arrow[dl, shift left] \\ & I_2 \arrow[ul] \arrow[ur, shift left] \arrow[loop below] & \end{tikzcd}$$

This transition graph does not indicate a horseshoe.

However, the induced graph of f^2-covering relations is

$$\begin{tikzcd} I_0 \arrow[dr, shift left] & & I_1 \arrow[ll] \arrow[loop right] \arrow[dl, shift left] \\ & I_2 \arrow[ul, shift left] \arrow[ur, shift left] \arrow[loop below] & \end{tikzcd}$$

This graph has a subgraph (the I_1, I_2 section) ~~with~~ that indicates f^2 has a horseshoe.

Hence f is chaotic.

b) Suppose $x_1 < x_0 < x_2 < x_3$
Let $I_0 = [x_1, x_0]$, $I_1 = [x_0, x_2]$, $I_2 = [x_2, x_3]$.

Then
$$f(I_0) \supseteq [x_1, x_2] = I_0 \cup I_1$$
$$f(I_1) \supseteq [x_1, x_3] = I_0 \cup I_1 \cup I_2$$
$$f(I_2) \supseteq [x_0, x_3] = I_1 \cup I_2$$

Hence the transition graph is

$$G \quad \begin{tikzcd} I_0 \arrow[rr, shift left] & & I_1 \arrow[ll, shift left] \arrow[loop right] \arrow[dl, shift left] \\ & I_2 \arrow[ur, shift left] \arrow[loop below] & \end{tikzcd}$$

This graph indicates that f has two horseshoes, and so is chaotic.

c) Suppose $x_3 < x_1 < x_2 < x_0$
Let $I_0 = [x_3, x_1]$, $I_1 = [x_1, x_2]$, $I_2 = [x_2, x_0]$

Then
$$f(I_0) \supseteq [x_2, x_0] = I_2.$$
$$f(I_1) \supseteq [x_3, x_2] = I_0 \cup I_1$$
$$f(I_2) \supseteq [x_3, x_1] = I_0$$

Hence the transition graph is

$$\begin{tikzcd} I_0 \arrow[d, shift left] & I_1 \arrow[l] \arrow[loop right] \\ I_2 \arrow[u, shift left] & \end{tikzcd}$$

This graph does not show a horseshoe. furthermore, $f^n(I_0)$ cannot cover for any n, so $\{I_0, I_1\}$ cannot form a horseshoe for any n. furthermore, for any n, either $f^n(I_0) \to I_2$ and $f^n(I_2) \to I_0$ <u>or</u> $f^n(I_0) \to I_0$ and $f^n(I_2) \to I_2$, but not <u>both</u>. So $\{I_0, I_2\}$ cannot form a horseshoe for any n. Clearly $\{I_1, I_2\}$ cannot form a horseshoe for any n either.

Hence f is not chaotic.

d). Suppose $x_1 < x_2 < x_0 < x_3$
Let $I_0 = [x_1, x_2]$, $I_1 = [x_2, x_0]$, $I_2 = [x_0, x_3]$.

Then
$$f(I_0) \supseteq I_1 \cup I_2$$
$$f(I_1) \supseteq I_1 \cup I_2 \cup I_3$$
$$f(I_2) \supseteq I_0 \cup I_1.$$

Hence the transition graph is

$$\begin{tikzcd} I_0 \arrow[rr, shift left] \arrow[dr, shift left] & & I_1 \arrow[ll, shift left] \arrow[loop right] \arrow[dl, shift left] \\ & I_2 \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd}$$

This graph does not show a horseshoe. However, the induced graph for f^2 is

$$\begin{tikzcd} I_0 \arrow[loop left] \arrow[rr, shift left] \arrow[dr, shift left] & & I_1 \arrow[ll, shift left] \arrow[loop right] \arrow[dl, shift left] \\ & I_2 \arrow[ul, shift left] \arrow[ur, shift left] \arrow[loop below] & \end{tikzcd}$$

which has horseshoes everywhere. Hence F is chaotic.

ii) ~~3~~. a) Transition graph

$$\begin{tikzcd} I_0 \arrow[rr] & & I_1 \arrow[dl, shift left] \\ & I_2 \arrow[ul] \arrow[ur, shift left] \arrow[loop below] & \end{tikzcd}$$

Transition matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

A has characteristic polynomial

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - \lambda^2 - \lambda - 1 = 0$$

Hence by Cayley-Hamilton

$$A^3 - A^2 - A - I = 0$$

Let P_n be the number of points in period n orbits and N_q be the number of q cycles. Then

$$P_n = tr(A^n)$$

and $$P_n = \sum_{q|n} q N_q.$$

Hence
$$P_1 = tr(A) = 1$$
$$P_2 = tr(A^2) = tr\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} = 3$$
$$P_3 = tr(A^3) = tr(A^2) + tr(A) + 3 = 7$$
$$P_4 = tr(A^4) = tr(A^3) + tr(A^2) + tr(A) = 11$$
$$P_5 = tr(A^5) = tr(A^4) + tr(A^3) + tr(A^2) = 21$$

Hence
$$N_1 = P_1 = 1$$
$$2N_2 = P_2 - N_1 \Rightarrow N_2 = 1$$
$$3N_3 = P_3 - N_1 \Rightarrow N_3 = 2$$
$$4N_4 = P_4 - 2N_2 - N_1 \Rightarrow N_4 = 2$$
$$5N_5 = P_5 - N_1 \Rightarrow N_5 = 4.$$

Using the transition graph to identify allowable sequences, these n-cycles correspond to the following sequences in the $\Sigma_{3,A}$ symbol space under the action of σ_A:

1-cycle: (2)
2-cycles: (1 2)
3-cycles: (0 1 2), (1 2 2)
4-cycles: (0 1 2 2), (1 2 2 2)
5-cycles: (0 1 2 2 2), (1 2 2 2 2), (1 2 1 2 2), (0 1 2 1 2)

c) Transition graph

$$\begin{tikzcd} I_0 \arrow[dr, shift left] & I_1 \arrow[l] \arrow[loop right] \\ & I_2 \arrow[ul, shift left] \end{tikzcd}$$

Transition matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

A has characteristic polynomial

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Hence by Cayley-Hamilton

$$A^3 - A^2 - A + I = 0$$

Let P_n be number of points in period n orbits and N_q be number of q-cycles.

Then
$$P_1 = tr(A) = 1$$
$$P_2 = tr(A^2) = tr\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 3$$
$$P_3 = tr(A^3) = tr(A^2) + tr(A) - 3 = 1$$
$$P_{\cancel{3}4} = tr(A^4) = tr(A^3) + tr(A^2) - tr(A) = 3$$
$$\vdots$$

and
$$P_{2k-1} = 1$$
$$P_{2k} = 3 \qquad k = 1, 2, \ldots$$

Hence
$$N_1 = P_1 = 1$$
$$N_2 = \frac{P_2 - N_1}{2} = 1$$
$$N_3 = \frac{P_3 - N_1}{3} = 0$$

$$N_4 = \frac{P_4 - 2N_2 - N_1}{4} = 0$$

$$N_5 = \frac{P_5 - N_1}{5} = 0$$

In the $\Sigma_{3,A}$ symbol space these cycles are

1-cycle: (1)
2-cycle: (0 2).

Note - Some definitions of chaos require that periodic points are dense.

MA30060 – Problem Sheet 6 – Solutions.

1. $F: I \to \mathbb{R}$, $\quad x_{n+1} = F(x_n)$, $(n \bmod 7)$

$$x_4 < x_2 < x_0 < x_6 < x_1 < x_3 < x_5$$

$I_0 = [x_4, x_2]$, $\quad I_1 = [x_2, x_0]$

$I_2 = [x_0, x_6]$, $\quad I_3 = [x_6, x_1]$

$I_4 = [x_1, x_3]$, $\quad I_5 = [x_3, x_5]$.

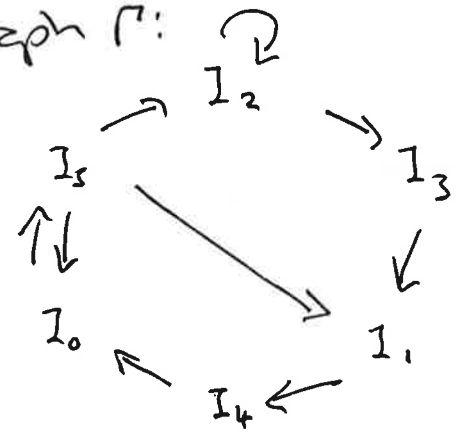
F-covering relations

$F(I_0) \supseteq [x_3, x_5] = I_5$, $\quad F(I_1) \supseteq [x_1, x_3] = I_4$

$F(I_2) \supseteq [x_0, x_1] = I_2 \cup I_3$, $\quad F(I_3) \supseteq [x_2, x_0] = I_1$

$F(I_4) \supseteq [x_4, x_2] = I_0$, $\quad F(I_5) \supseteq [x_4, x_6] = I_0 \cup I_1 \cup I_2$.

Transition graph Γ:



b) F is semiconjugate to the SSFT σ_A where A is the transition matrix for F and the intervals I_i.

Hence F has at least as many periodic orbits as σ_A.

N-cycles can be identified by inspection of the transition graph Γ:

N=2

$I_0 I_5 \to I_0$

N=4

$I_5 I_1 I_4 I_0 \to I_5$

N=6

$I_5 I_1 I_4 I_0 I_5 I_0 \to I_5$

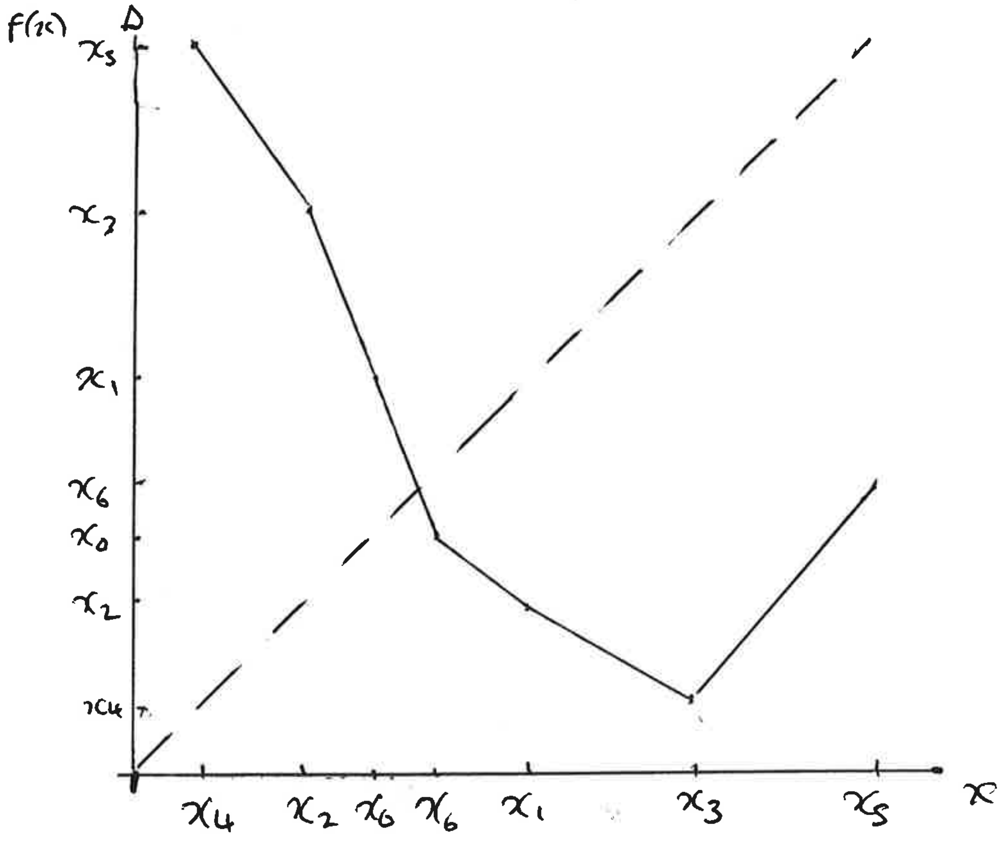
$N \geq 8$

$I_3 I_1 I_4 I_0 I_5 I_2^{N-6} \to I_3$ is a period N orbit in $\Sigma_{6,A}$.

b) By inspection, there are no closed paths of length 3 or length 5 in Γ. Hence F does not have a 3-cycle or a 5-cycle.

c) Construct a graph for F.

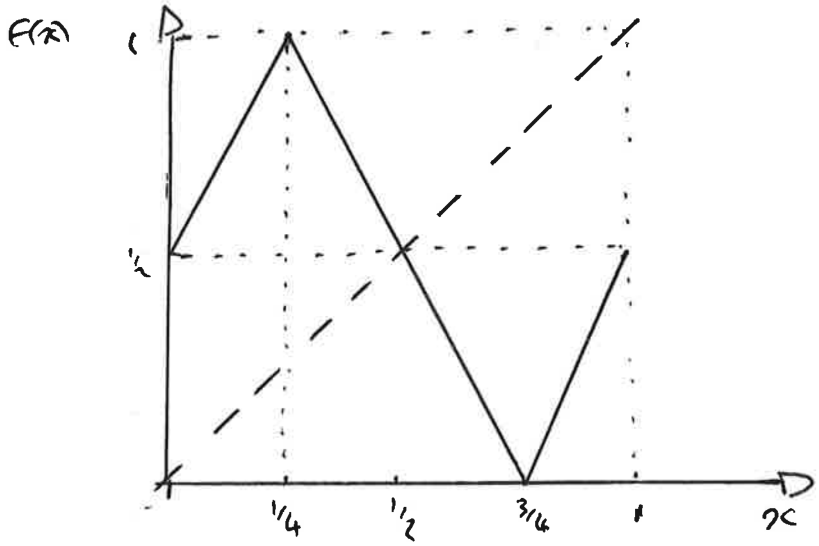
Simplest construction is piecewise linear



2. $f: [0,1] \to [0,1]$.

$$f(x) = \begin{cases} 2x + 1/2, & 0 \le x \le 1/4 \\ \frac{3}{2} - 2x, & 1/4 \le x \le 3/4 \\ 2x - \frac{3}{2} & 3/4 \le x \le 1. \end{cases}$$

<u>Sketch</u> <u>$f(x)$</u>



<u>Construct</u> <u>$f^2(x)$</u>
By observation

if $0 \le x \le 1/8$, $1/2 \le f(x) \le 3/4$, $f^2(x) = \frac{3}{2} - 2f(x)$
$= \frac{1}{2} - 2x$.

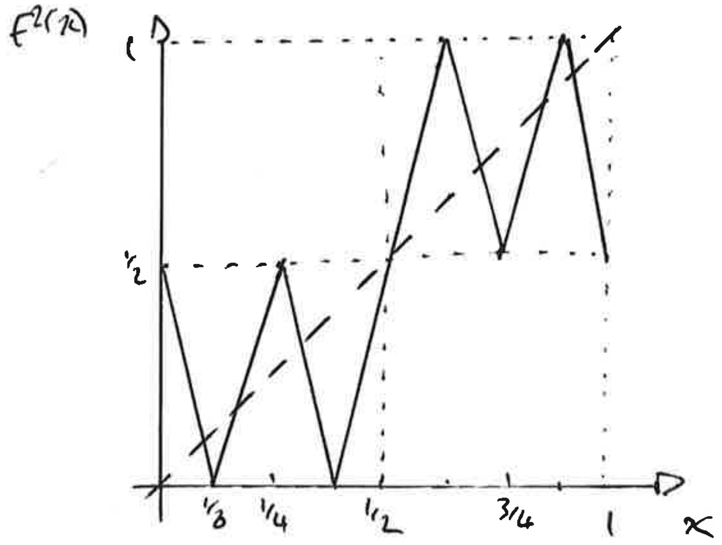
$1/8 \le x \le 1/4$, $3/4 \le f(x) \le 1$, $f^2(x) = 2f(x) - 3/2$
$= 4x - 1/2$

$1/4 \le x \le 3/8$, $3/4 \le f(x) \le 1$, $f^2(x) = 2f(x) - 3/2$
$= \frac{3}{2} - 4x$

$3/8 \le x \le 1/2$, $1/2 \le f(x) \le 3/4$, $f^2(x) = \frac{3}{2} - 2f(x)$
$= 4x - \frac{3}{2}$

and similarly for $[1/2, 5/8]$, $[5/8, 3/4]$, $[3/4, 7/8]$, $[7/8, 1]$.

Sketch $f^2(x)$



Even period orbits of f

Let $\quad I = [0, \tfrac{1}{2}], \quad I_0 = [0, \tfrac{1}{8}], \quad I_1 = [\tfrac{1}{8}, \tfrac{1}{4}]$

Then $\quad f^2(I_0) = I, \quad f^2(I_1) = I.$

Hence $\quad f^2$ has a horseshoe.

Hence $\quad f$ is chaotic.

In addition, f^2 has periodic orbits of every period. Hence f has periodic orbits of every even period.

Eg. If $\{x_1, x_2\}$ a 2-cycle under f^2 then
$f^2(x_1) = f(f(x_1)) = x_2, \quad f^2(x_2) = f(f(x_2)) = x_1$.
So $\{x_1, f(x_1), x_2, f(x_2)\}$ a 4-cycle under f.

From the sketch for f, $x = \tfrac{1}{2}$ is a fixed point for f.

Odd period orbits of f

By inspection of the graphs for f and f^2, if $x_0 \in [0, \tfrac{1}{2}]$ then $f^2(x_0) \in [0, \tfrac{1}{2}]$, so $f^{2k}(x_0) \in [0, \tfrac{1}{2}] \quad \forall\, k \geq 0$.

But $f(x_0) \in [1/2, 1]$. So $f(f^{2k}(x_0)) = f^{2k+1}(x_0) \in [1/2, 1]$ $\forall\, k \geq 0$.

If x_0 is in an orbit with odd period then, for some $k \geq 1$, $x_0 = f^{2k+1}(x_0)$.

Then $x_0 \in [0, 1/2] \cup [1/2, 1] \Rightarrow x_0 = 1/2$. So x_0 is the fixed point of f.

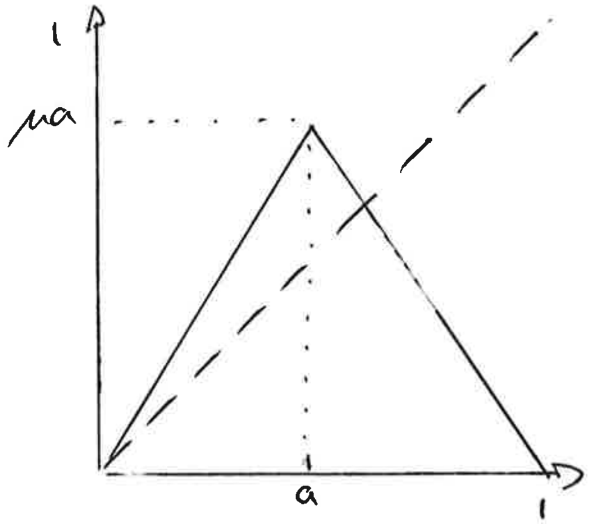
Hence there are no points on $2k+1$-cycles for any $k \geq 1$. ie there are no odd period N-cycles.

3. <u>Skewed tent map</u>

$$f(x) = \begin{cases} \mu x, & 0 \leq x \leq a \\ \frac{\mu a}{1-a}(1-x), & a \leq x \leq 1, \end{cases}$$

where $0 < a < 1$.

a)



f maps $[0,1]$ into itself if $0 \leq \mu a \leq 1$
$\Rightarrow$ $0 \leq \mu \leq 1/a$.

<u>fixed points</u>

$x=0$ is always a fixed point.

A non-trivial fixed point will exist as long as the initial gradient of f is above 1, i.e. if $\mu > 1$.

The non-trivial fixed point occurs on the downward section of f_1 at $x = \frac{\mu a(1-x)}{1-a}$

$$\Rightarrow (1-a)x = \mu a - \mu a x \Rightarrow x = \frac{\mu a}{1-a+\mu a}.$$

Gradient at non-trivial fixed point is, by inspection, $\frac{-\mu a}{1-a}$.

So non-trivial fixed point stable if

$$\frac{-\mu a}{1-a} > -1 \Rightarrow -\mu a > a-1$$
$$\Rightarrow \mu < \frac{1}{a} - 1.$$

Hence non-trivial fixed point exists and is stable if

$$1 < \mu < \frac{1}{a} - 1.$$

b) <u>Construct f^2</u>

Define
$$f(x) = \begin{cases} f_L(x) = \mu x, & 0 \le x \le a \\ f_R(x) = \frac{\mu a}{1-a}(1-x), & a \le x \le 1. \end{cases}$$

Then
$$f^2(x) = \begin{cases} f_L \circ f_L(x), & 0 \le x \le a \ \& \ 0 \le f_L(x) \le a \\ f_R \circ f_L(x), & 0 \le x \le a \ \& \ a \le f_L(x) \le 1 \\ f_R \circ f_R(x), & a \le x \le 1 \ \& \ a \le f_R(x) \le 1 \\ f_L \circ f_R(x), & a \le x \le 1 \ \& \ 0 \le f_R(x) \le a \end{cases}$$

So

$$f^2(x) = \begin{cases} \mu^2 x & 0 \le x \le a \ \ \& \ \ 0 \le x \le \frac{a}{\mu} \quad (1) \\ \frac{\mu a}{1-a}(1-\mu x) & 0 \le x \le a \ \ \& \ \ \frac{a}{\mu} \le x \le \frac{1}{\mu} \quad (2) \\ \frac{\mu a}{1-a}\left(1 - \frac{\mu a}{1-a}(1-x)\right) & a \le x \le 1 \ \ \& \ \ a \le \frac{\mu a}{1-a}(1-x) \le 1 \quad (3) \\ \frac{\mu^2 a}{1-a}(1-x) & a \le x \le 1 \ \ \& \ \ 0 \le \frac{\mu a}{1-a}(1-x) \le a \quad (4) \end{cases}$$

Given $0 < a < 1$, $\mu > 1$ and $\mu a \le 1$, the constraints simplify such that:

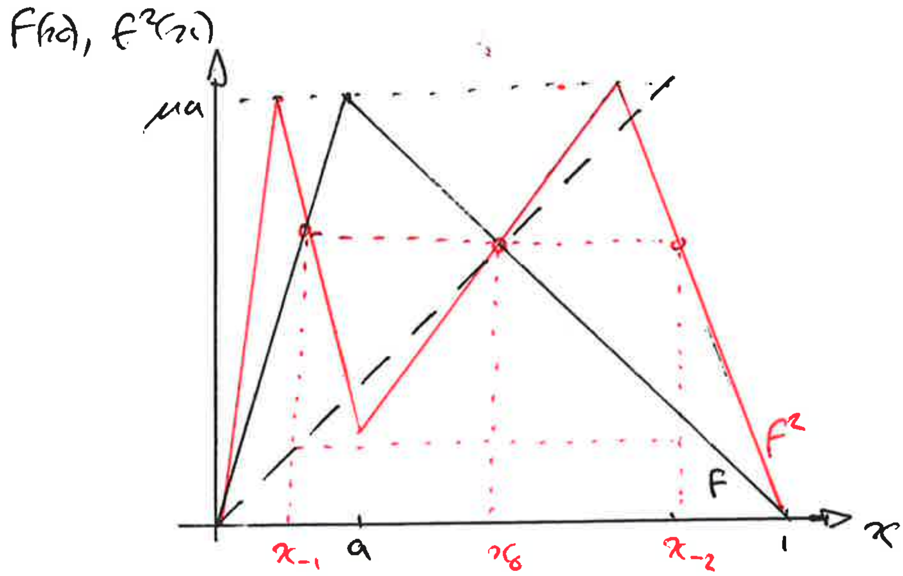
$$f^2(x) = \begin{cases} \mu^2 x, & 0 \le x \le a/\mu \\ \frac{\mu a}{1-a}(1-\mu x), & \frac{a}{\mu} \le x \le a \qquad (\mu a \le 1) \\ \frac{\mu a}{1-a}\left(1 - \frac{\mu a}{1-a}(1-x)\right) & a \le x \le 1 - \frac{1-a}{\mu} \qquad (*) \\ \frac{\mu^2 a}{1-a}(1-x) & 1 - \frac{1-a}{\mu} \le x \le 1 \end{cases}$$

$$(*) \quad a \le \frac{\mu a}{1-a}(1-x) \Rightarrow 1-a \le \mu(1-x)$$
$$\Rightarrow a + \mu - 1 \ge \mu x$$
$$\Rightarrow x \le \frac{a+\mu-1}{\mu} = 1 - \frac{(1-a)}{\mu} < 1.$$

<u>Sketch</u> <u>$f^2(x)$</u>

endpoints of each linear section are

$$f^2(0) = 0$$
$$f^2(a/\mu) = a\mu$$
$$f^2(a) = \frac{\mu a}{1-a}(1-\mu a) > 0$$
$$f^2\left(1 - \frac{1-a}{\mu}\right) = \mu a$$
$$f^2(1) = 0.$$



x_{-1} preimage of x_0 under f (and hence f^2):
So

$$f(x_{-1}) = x_0 \Rightarrow \mu x_{-1} = \frac{\mu a}{1-a+\mu a}$$

$$\Rightarrow x_{-1} = \frac{a}{1-a+\mu a}.$$

c) ~~There is~~ $f^2(x)$ contains an (upside down) skewed tent map on $[x_{-1}, x_0] \doteq I_L$

Let $x \in I_L$. Then $f^2(x) \geq f^2(a)$, and $f^2(x) \leq f^2(x_{-1}) = f^2(x_0) = x_0$.

Hence $f^2(I_L) \supset I_L$ if $f^2(a) < x_{-1}$.

Then f^2 has a horseshoe since $I_{L1} = [x_{-1}, a]$ and $I_{L2} = [a, x_0]$ are subintervals of I_L such that $f(I_{L1}) \supset I_L$ and $f(I_{L2}) \supset I_L$.

Now $f^2(a) = \frac{\mu a}{1-a}(1-\mu a) < x_{-1} = \frac{a}{1-a+\mu a}$

$$\text{if} \quad \mu(1-\mu a)(1-a+\mu) < 1-a$$
$$\Rightarrow \quad \mu(1-\mu a - a + \mu a^2 + \mu a - \mu^2 a^2) < 1-a$$
$$\Rightarrow \quad \mu(1-a) + \mu^2 a^2 - \mu^3 a^2 - (1-a) < 0$$
$$\Rightarrow \quad \mu^3 - \mu^2 - \mu\left(\frac{1-a}{a^2}\right) + \frac{1-a}{a^2} > 0$$
$$\Rightarrow \quad (\mu-1)\left(\mu^2 - \frac{1-a}{a^2}\right) > 0$$
$$\Rightarrow \quad \mu^2 > \frac{1-a}{a^2}.$$

Hence f is chaotic if $\mu^2 > \frac{1-a}{a^2}$.

$f^2(x)$ contains another skewed tent map on $[x_0, x_{-2}] \doteq I_R$.

Let $x \in I_R$. Then $f^2(x) \geq f^2(x_0) = f^2(x_{-2}) = x_0$, and $f^2(x) \leq f^2\left(1 - \frac{1-a}{\mu}\right) = \mu a$.

Hence f^2 has a horseshoe on I_R if $f^2\left(1-\frac{1-a}{\mu}\right) = \mu a > x_{-2}$.

After some algebra, $x_{-2} = \dfrac{\mu a(1-a+\mu a) - a(1-a)}{\mu a(1-a+\mu a)}$

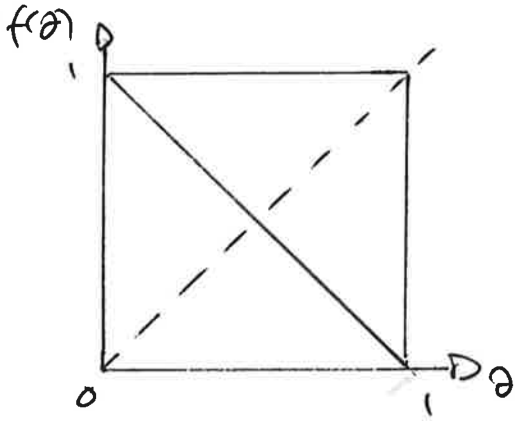
and $\mu a > x_{-2}$ if $(\mu-1)\left(\mu^2 - \frac{1-a}{a^2}\right)$, as above.

Hence f^2 has horseshoes on both I_L and I_R, or neither.

MA30060 - PROBLEM SHEET 7 - SOLUTIONS

1. a) $f(\theta) = 1-\theta.$

i)



ii) A lift $F:\mathbb{R}\to\mathbb{R}$ satisfies $f\circ\pi(x) = \pi\circ F(x)$.

so

$$f\circ\pi(x) = \pi\circ F(x)$$
$$\Rightarrow \quad 1-\pi(x) = \pi\circ F(x)$$
$$\Rightarrow \quad 1-x \bmod 1 = F(x) \bmod 1$$
$$\Rightarrow \quad F(x) = k-x \quad \text{for any } k\in\mathbb{Z}.$$

iii)
$$\begin{aligned}\deg(f) &= F(x+1)-F(x)\\ &= k-(x+1)-(k-x)\\ &= -1.\end{aligned}$$

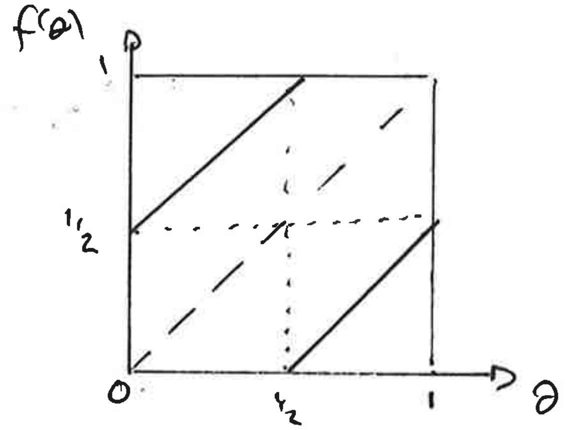
iv) Let $z = e^{2\pi i\theta}$ and $\omega(z) = e^{2\pi i f(\theta)} = e^{2\pi i(1-\theta)}$

Then $\omega(z) = e^{-2\pi i\theta} = \bar{z}.$

b)

$f(\theta) = \theta + \frac{1}{2} \bmod 1.$

i)



ii) Lifts:

$$\tfrac{1}{2} + x \bmod 1 = f(x) \bmod 1$$

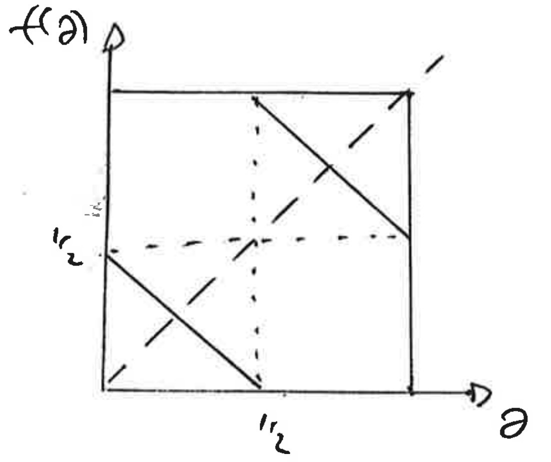
$$\Rightarrow \quad f(x) = k + x + \tfrac{1}{2} \quad \text{for any } k \in \mathbb{Z}.$$

iii) $$\deg(f) = f(x+1) - f(x) = 1.$$

iv) $$\omega(z) = e^{2\pi i(\theta + 1/2)} = -z.$$

c) $$f(\theta) = 1/2 - \theta \pmod 1$$

i)



ii) Lifts :

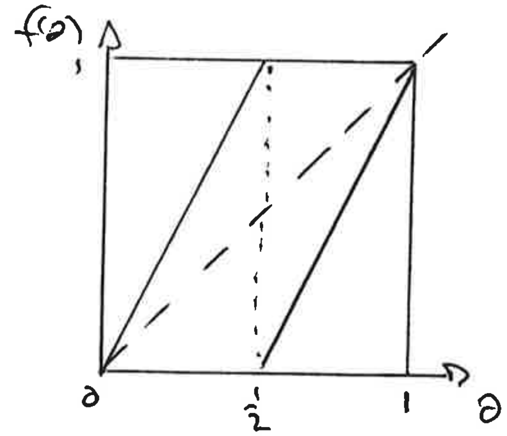
$$\tfrac{1}{2} - x \mod 1 = f(x) \mod 1$$
$$\Rightarrow \quad f(x) = k + \tfrac{1}{2} - x \quad \text{for any } k \in \mathbb{Z}.$$

iii)
$$\deg(f) = F(x+1) - F(x)$$
$$= -1.$$

iv)
$$w(z) = e^{2\pi i(\frac{1}{2} - \theta)} = e^{-2\pi i(\theta - \frac{1}{2})} = -\bar{z}.$$

d)
$$f(\theta) = m\theta \mod 1 \quad \text{for } m \in \mathbb{Z}.$$

i) for $m = 2$



ii) Lifts

$$m(x \mod 1) = f(x) \mod 1$$
$$\Rightarrow \quad f(x) = k + mx \quad \text{for any } k \in \mathbb{Z}.$$

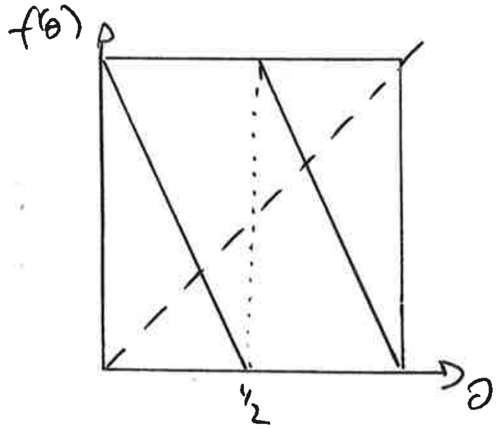
iii)
$$\deg(f) = f(x+1) - f(x)$$
$$= m$$

for $m = 2$ above, $\deg(f) = 2$.

iv)
$$w(z) = e^{2\pi i m\theta} = z^m.$$

e) $f(\theta) = -m\theta \mod 1$

i) for $m=2$



ii) Lifts:

$$-m(x \mod 1) = F(x) \mod 1$$

$$\Rightarrow \quad F(x) = k - mx \quad \text{for any } k \in \mathbb{Z}.$$

iii) $$\deg(f) = F(x+1) - F(x)$$
$$= -m$$

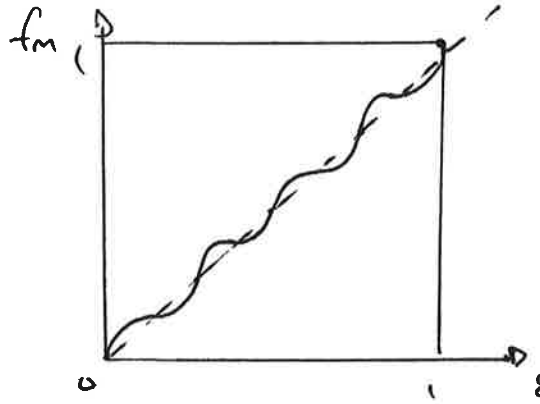
iv) $$w(z) = e^{-2\pi i m\theta} = \bar{z}^m.$$

2. $f_m : S^1 \to S^1$ circle map with lift
$F_m : \mathbb{R} \to \mathbb{R}$, $f_m(x) = x + \frac{1}{2\pi m}\sin(2\pi m x)$

a)

$$\deg(f_m) = F_m(x+1) - F_m(x)$$
$$= x + 1 + \frac{1}{2\pi m}\sin(2\pi m x + 2\pi m) - x - \frac{1}{2\pi m}\sin(2\pi m x)$$
$$= 1 \quad \text{since } m \in \mathbb{Z}.$$

b) for $m=4$, f_m looks like



c) within $[0,1]$ fixed points satisfy

$$f_m(x) = x + \frac{1}{2\pi m} \sin(2\pi m x) = x$$

$$\Rightarrow \quad \sin(2\pi m x) = 0$$

$$\Rightarrow \quad 2\pi m x = n\pi, \quad n \in \mathbb{Z}$$

$$\Rightarrow \quad x = 0, \frac{1}{2m}, \frac{2}{2m}, \ldots \frac{2(m-1)}{2m}.$$

Hence there are $2m$ distinct fixed points in S^1.

d) f_m and f_n have a different number of fixed points if $m \neq n$. Hence they cannot be topologically conjugate.

e)

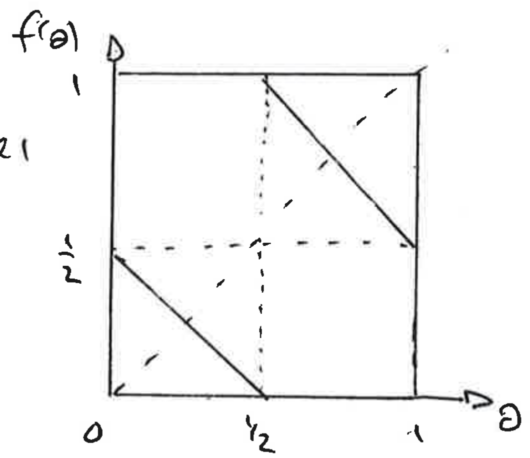
$$\rho_0(F, 0) = \lim_{n\to\infty} \frac{F_m^n(0)}{n} = \lim_{n\to\infty} \frac{0}{n} = 0.$$

So $\rho(f) = 0$.

Note that this must be true since f has fixed points.

3. a)

$f(\theta) = \frac{1}{2} - \theta \mod 1$



Lifts: $F(x) = k + \frac{1}{2} - x, \quad k \in \mathbb{Z}.$

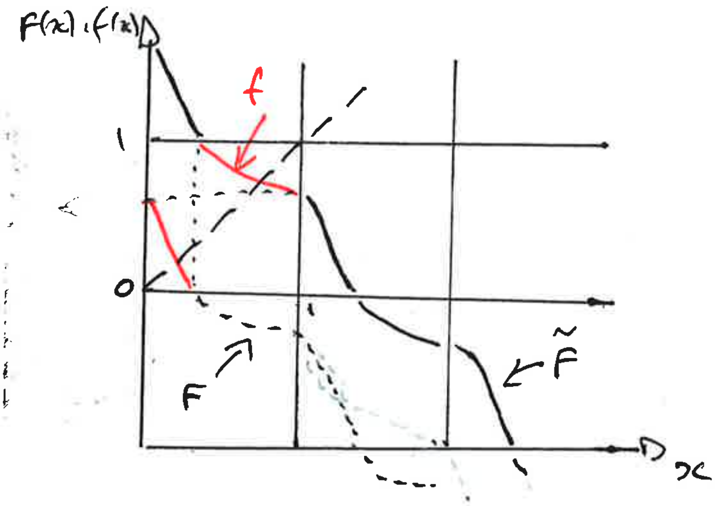
So $F(x) > F(y) \iff x < y$ and f is an order-reversing homeomorphism.

By observation, f has two fixed points (at $\theta = 1/4, 3/4$).

b) Let $f: S^1 \to S^1$ be an orientation-reversing homeomorphism with lift $F: \mathbb{R} \to \mathbb{R}$.

Then $F(x+1) = F(x) - 1$ since f ORH.

For convenience let F be the lift such that $0 \leq F(0) < 1$.



Let $G(x) = f(x) - x$ on $[0,1]$.

Then $G(0) = f(0) \geq 0$
$G(1) = f(1) - 1 \leq 0$ since $f(1) < f(0)$
$\Rightarrow f(1) - 1 < f(0) - 1 < 0$

Hence, by the Intermediate Value Theorem, there exists p such that $0 \leq p < 1$ and $G(p) = 0 \Rightarrow f(p) = p$.

Now let $\tilde{F}$ be the lift $F(x) + 1$.

Then $\tilde{F}(p) = f(p) + 1 = p + 1 > 0$ since $0 \leq p < 1$.
$\tilde{F}(1) = F(1) + 1 = f(0) - 1 + 1 = f(0) < 1$.

So $\tilde{G}(x) = \tilde{F}(x) - x$ satisfies

$$\tilde{G}(p) = \tilde{F}(p) - p = 1 > 0$$
$$\tilde{G}(1) = \tilde{F}(1) - 1 < 0$$

Hence, by the Intermediate Value Theorem, there exists q such that $\tilde{F}(q) = q$, and $p < q < 1$.
Both p and q are therefore fixed points of f.

MA30060 - PROBLEMS 8 - SOLUTIONS

1. a) Rigid rotation $r_\beta(x) = x+\beta \mod 1$

Lift $R_\beta(x) = x+\beta+k, \quad k \in \mathbb{Z}.$

So $$\rho_0(R_\beta, 0) = \lim_{n\to\infty} \frac{R_\beta^n(0)}{n}$$

$$= \lim_{n\to\infty} \frac{n(\beta+k)}{n} = \beta+k.$$

Hence rotation number $\rho(r_\beta) = \beta+k \mod 1$
$= \beta \mod 1$
$= \beta$ if $0<\beta<1$.

b) f an ORH with lift F.

i) $$f^2 \circ \pi(x) = f\circ f\circ \pi(x)$$
$$= f\circ \pi \circ F(x) \qquad (F \text{ a lift of } f)$$
$$= \pi \circ F\circ F(x)$$
$$= \pi \circ F^2(x)$$

Hence F^2 is a lift of f^2.

ii) for any ORH of the circle
$F(x+1) = F(x)-1.$

So $F(y) = F(y-1)-1$
$\Rightarrow F(y-1) = F(y)+1, \quad \forall y \in \mathbb{R}.$

iii) $$\deg(f^2) = F^2(x+1) - F^2(x)$$
$$= F(F(x+1)) - F^2(x)$$
$$= F(F(x)-1) - F^2(x) \qquad \text{since } f \text{ ORH}$$
$$= F^2(x)+1 - F^2(x)$$
$$= +1.$$

$\deg(f^2) = +1$ is a necessary but not sufficient condition for f to be an OPH.

In addition: f is a continuous bijection $\Rightarrow$ f^2 is a continuous bijection.

And: $f(x) > f(y) \Leftrightarrow x < y$ (f ORH)
$\Rightarrow f^2(x) > f^2(y) \Leftrightarrow f(x) < f(y)$
$\Leftrightarrow x > y$.

Hence f^2 is an OPH.

iv) f is an ORH $\Rightarrow$ f has exactly two fixed points.

Let x_0 be a fixed point of f.
Then $x_0 = f(x_0) = f^2(x_0)$.

Hence x_0 is a fixed point of f^2.

Let $g = f^2$. Then x_0 is a fixed point of g.

Let ~~Ĝ~~ G be a lift of g such that $G(x_0) = x_0$.

Then $\rho_0(G, x_0) = \lim_{n\to\infty} \frac{G^n(x_0) - x_0}{n} = \frac{x_0 - x_0}{n} = 0$

Hence $\rho(f^2) = \rho(g) = \rho_0(G) \bmod 1 = 0$.

2. f an OPH of the circle with lift F.
x_0 periodic with least period q.

i)
know $f^q(x_0) = x_0$ and $f^j(x_0) \neq x_0$ for $j = 1..q-1$.

know F^q is a lift for f^q.

Hence $f^q \cdot \pi(x) = \pi \cdot F^q(x)$, $\pi(x_0) = x_0$.

So
$$\begin{aligned} x_0 &= f^q \cdot \pi(x_0) && (\pi(x_0) = x_0,\ f(x_0) = x_0) \\ &= \pi \cdot F^q(x_0) && (F^q \text{ lift for } f^q) \\ &= F^q(x_0) \bmod 1 && (\text{by definition}) \end{aligned}$$

Hence $F^q(x_0) = x_0 + p$ for some $p \in \mathbb{Z}$.

ii) First, consider $F^{2q}(x_0)$:

$$\begin{aligned} F^{2q}(x_0) &= F^q(x_0 + p) && (F^{2q}(x_0) = F^q \cdot F^q(x_0)) \\ &= F^q(x_0) + p && (\text{since } F^q \text{ an OPH}) \\ &= x_0 + 2p \end{aligned}$$

Now, suppose result holds for $n-1$:

$$F^{(n-1)q}(x_0) = x_0 + (n-1)p.$$

Then
$$\begin{aligned} F^{nq}(x_0) &= F^q F^{(n-1)q}(x_0) \\ &= F^q(x_0 + (n-1)p) \\ &= F^q(x_0) + (n-1)p \\ &= x_0 + np. \end{aligned}$$

Hence result holds for all n by induction

iii) Then $$\rho_0(F) = \lim_{n\to\infty} \frac{F^{nq}(x_0) - x_0}{nq}$$
$$= \lim_{n\to\infty} \frac{x_0 + np - x_0}{nq} = \frac{p}{q}.$$

Hence $\rho(f) = \frac{p}{q} \bmod 1$, which is rational.

3.

$\rho(f_{k,\omega}) = \frac{1}{2}$ if and only if $f_{k,\omega}(x)$ has a least period 2 orbit.

$f_{k,\omega}(x)$ has a least period 2 orbit if, for the lift $F_{k,\omega}(x)$, $F^2_{k,\omega}(x) = x+1$

So, a period 2 orbit exists, and $\rho(f_{k,\omega}) = \frac{1}{2}$ when a solution (for x) exists for

$$F^2_{k,\omega}(x) = x+1$$
$$\Rightarrow F_{k,\omega}\left(x + \omega + \frac{k}{2\pi}\sin(2\pi x)\right) - x = 1$$
$$\Rightarrow x + \omega + \frac{k}{2\pi}\sin(2\pi) + \omega + \frac{k}{2\pi}\sin\left(2\pi\left(x + \omega + \frac{k}{2\pi}\sin(2\pi x)\right)\right) - x = 1$$

Set $\omega = \frac{1}{2} + \varepsilon$, $\varepsilon \ll 1$.

Then, cancelling the x terms,

$$1 + 2\varepsilon + \frac{k}{2\pi}\sin(2\pi x) + \frac{k}{2\pi}\sin\left(2\pi x + \pi + 2\pi\varepsilon + k\sin(2\pi x)\right) = 1$$

$$\Rightarrow 4\pi\varepsilon + k\sin(2\pi x) + k\Big(\sin(2\pi x + \pi)\cos(2\pi\varepsilon + k\sin(2\pi x))$$
$$+ \cos(2\pi x + \pi)\sin(2\pi\varepsilon + k\sin(2\pi x)\Big) = 0$$

$$\Rightarrow \quad 4\pi\varepsilon + k\sin(2\pi x) - k\sin(2\pi x)\cos(2\pi\varepsilon + k\sin(2\pi x)) - \cos(2\pi x)\sin(2\pi\varepsilon + k\sin(2\pi x)) = 0$$

$$\Rightarrow \quad 4\pi\varepsilon + k\sin(2\pi x) - k\sin(2\pi x)\left(1 - \tfrac{1}{2}(2\pi\varepsilon + k\sin(2\pi x))^2 + \ldots\right) - k\cos(2\pi x)\left(2\pi\varepsilon + k\sin(2\pi x) - \ldots\right) = 0$$

$$\Rightarrow \quad 4\pi\varepsilon + O(\varepsilon^2 k) + O(\varepsilon k^2) + O(k^3) + O(\varepsilon k) + \ldots - k^2\sin(2\pi x)\cos(2\pi x) + O(k|\varepsilon, k|^3) = 0$$

So, to leading order,

$$4\pi\varepsilon - \frac{k^2}{2}\sin(4\pi x) = 0$$

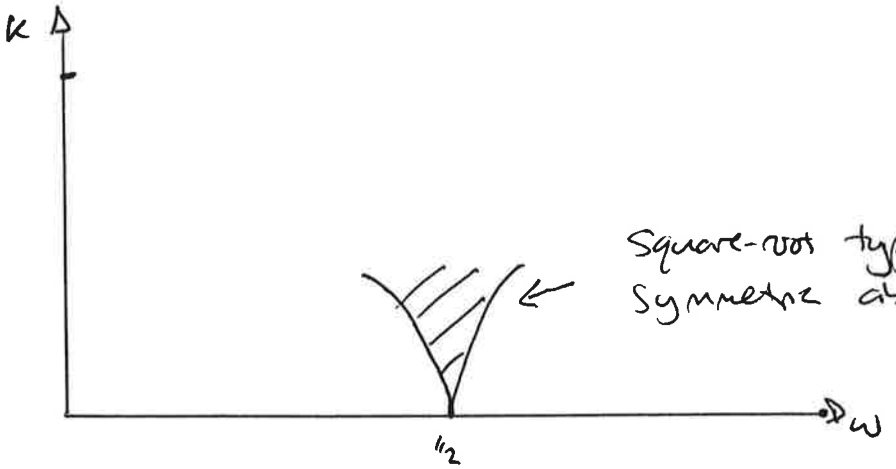
which has solutions when

$$\left|\frac{8\pi\varepsilon}{k^2}\right| < 1$$

$$\Rightarrow \quad |\varepsilon| < \frac{k^2}{8\pi}.$$

This corresponds to the region of the $(\omega - k)$ plane given by

$$|\omega - \tfrac{1}{2}| < \frac{k^2}{8\pi}.$$



Square-root type curves symmetric about $\omega = \frac{1}{2}$.