CHAPTER 3 CONTINUED.

LECTURE 3

Def'n 3.6 (Reversed curve $-C$)

Given curve $C \subset \mathbb{R}^3$ with paramet. $\underline{r}(t)$ for $t \in [a,b]$, then $-C$ is the curve given by. $\tilde{\underline{r}}(t)$ with $\tilde{\underline{r}}(t) = \underline{r}(-t)$

so $\hat{\underline{r}}(t)$ runs backwards.

§3.2 Line integrals of scalars.

Def'n 3.7 (Line integral of scalar)

Let C be given by $\underline{r}(t)$, $t \in [a,b]$.

We define the line integral of $f(t)$ by

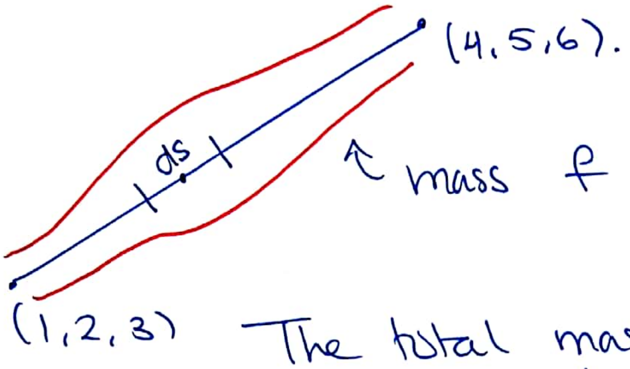
$$\int_C f \cdot ds \equiv \int_a^b f(\underline{r}(t)) |\underline{r}'(t)| \, dt$$

We also write $\oint_C f ds$ if C is closed.

Example 3.8. The mass per unit length along a curve given by the straight line from $(1,2,3)$ to $(4,5,6)$ is

$$f(x,y,z) = x + y + z.$$

What is the total mass?



mass $f = x + y + z$

The total mass is

$$M = \int_C f \, ds = \int_{t=a}^{t=b} f(\underline{r}(t)) \, \|\underline{r}'(t)\| \, dt.$$

We parameterise by

$$\underline{r}(t) = (1,2,3) + (4-1, 5-2, 6-3)\,t$$
$$= (1,2,3) + (3,3,3)\,t$$

note that $t = 0 \Rightarrow \underline{r}(0) = (1,2,3)$

$t = 1 \Rightarrow \underline{r}(1) = (4,5,6)$

$$M = \int_{t=0}^{t=1} \left[(1+3t) + (2+3t) + (3+3t)\right] \cdot \times \left|(3,3,3)\right| \cdot dt = 3\sqrt{3}\left(6 + \frac{9}{2}\right).$$

Lemma 3.9 gives basic properties of linearity, additivity, and reversability. [See problem set].

§ 3.3. Line of vector fields.

Def'n 3.10. (Vector of scalar integrals)

Let C be suff. nice curve and $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then

$$\int_C \underline{F} \, ds = \int_C \begin{pmatrix} F_1(\underline{x}) \\ F_2(\underline{x}) \\ F_3(\underline{x}) \end{pmatrix} ds = \begin{pmatrix} \int F_1 \, ds \\ \int F_2 \, ds \\ \int F_3 \, ds \end{pmatrix}$$

$$\underline{x} = (x, y, z)$$

* not really used.

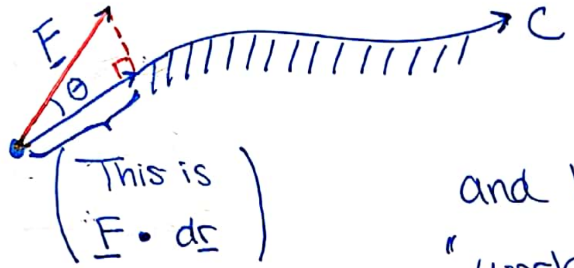
Really our focus is on what we call <u>work integrals</u>

Def'n 3.11 (Work integral of vector field)

Let C be suff. nice curve with $\underline{r}(t)$, $t \in [a, b]$ and let $\underline{F}$ be suff. nice vector field. The work integral is

$$\int_C \underline{F} \cdot d\underline{r} \equiv \int_{t=a}^{b} \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t) \, dt.$$

note that $\underline{F} \cdot d\underline{r}$ is the component of the field along the tangent to the curve since $\underline{F} \cdot d\underline{r} = |\underline{F}||d\underline{r}|\cos\theta$.



and hence $\underline{F} \cdot d\underline{r}$ is the "work" done at the given point.

Example 3.12 Calculate the work integral for $\underline{F} = (3xy, -5z, 10x)$ and a curve $\underline{r}(t) = (t^2+1, 2t^2, t^3)$, $t \in [1, 2]$.

$$\int_C \underline{F} \cdot d\underline{r} = \int_{t=1}^{t=2} \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt$$

$$= \int_1^2 \begin{pmatrix} 3(t^2+1)(2t^2) \\ -5t^3 \\ 10(t^2+1) \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 4t \\ 3t^2 \end{pmatrix} dt$$

$$= \cdots = 303.$$

Lemma 3.13 : This lemma covers the 3 basic properties of work integrals:

1) Linearity : $\int_C (\lambda \underline{F} + \mu \underline{G}) \cdot d\underline{r} = \lambda \int_C \underline{F} \cdot d\underline{r} + \mu \int_C \underline{G} \cdot d\underline{r}$

λ, μ scalar constants

2) Additivity: If $C = C_1 \cup C_2$ then

$$\int_C \underline{F} \cdot d\underline{r} = \int_{C_1} \underline{F} \cdot d\underline{r} + \int_{C_2} \underline{F} \cdot d\underline{r}$$

3) Non-independence of direction:

$$\int_{-C} \underline{F} \cdot d\underline{r} = - \int_C \underline{F} \cdot d\underline{r}$$