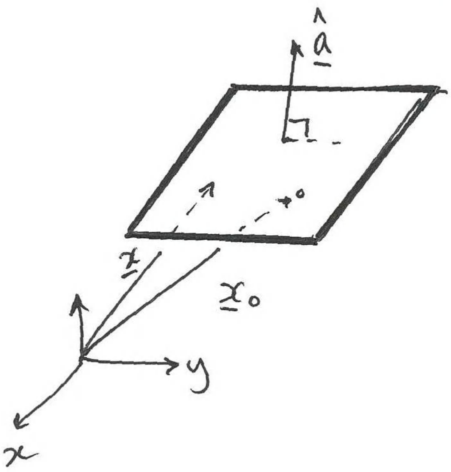
PROBLEM SET #2: SHORTENED SOL'NS.

22 Feb. 2018
PHT

#1. $F(\underline{x}) = \underline{a} \cdot \underline{x} - c = 0$

$$\hat{\underline{n}} = \frac{\nabla F}{|\nabla F|} = \frac{\underline{a}}{|\underline{a}|}$$

i.e. $\underline{a}$ is oriented along normal

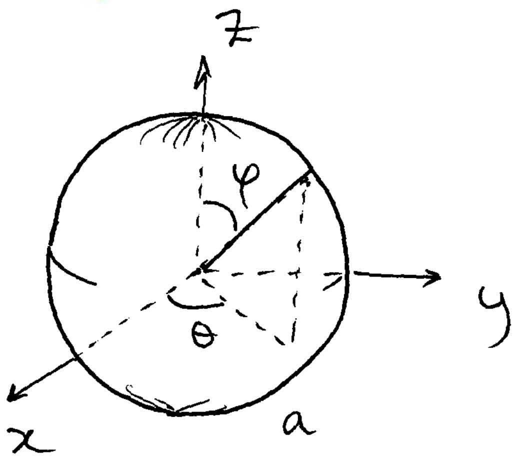


note we can write $\underline{a} \cdot (\underline{x} - \underline{x}_0) = 0$

where $\underline{a} \cdot \underline{x}_0 = c$ and $\underline{x}_0$ is point to the surface.

b) $\underline{r} = (a\cos\theta \sin\varphi,\ a\sin\theta \sin\varphi,\ a\cos\varphi)$

$0 \le \theta < 2\pi,\quad 0 \le \varphi < \pi$



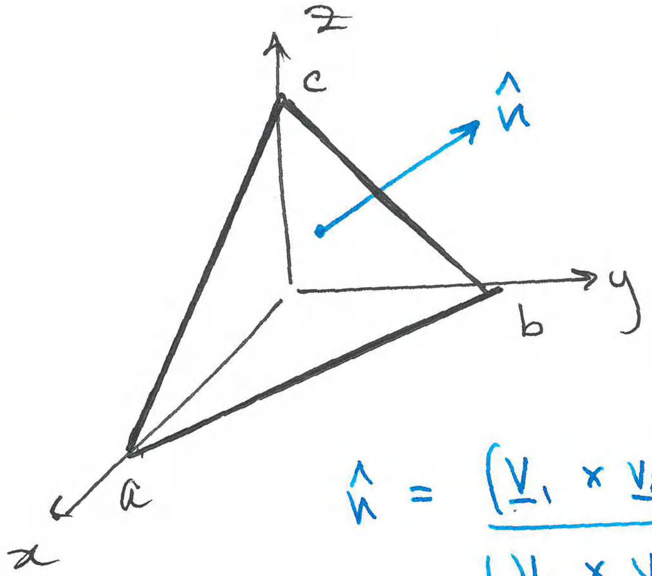
Normal is simply radial vector

$$\hat{\underline{n}} = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}}$$

To derive, consider $F = x^2+y^2+z^2-a^2$

and take $\frac{\nabla F}{|\nabla F|}$

(c)

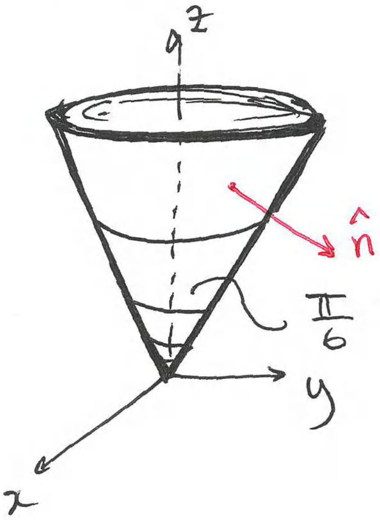


multiply by (-1) to get <u>outer</u> norm

$$\hat{n} = \frac{(\underline{V}_1 \times \underline{V}_2)}{|\underline{V}_1 \times \underline{V}_2|} = \frac{(+bc, +ac, +ab)}{\sqrt{(bc)^2 + (ac)^2 + (ab)^2}}$$

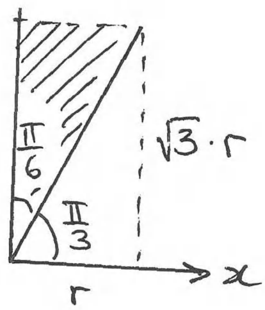
where $\underline{V}_1 = \underbrace{(a, -b, 0)}_{\overrightarrow{BA}}$ and $\underline{V}_2 = \underbrace{(0, -b, c)}_{\overrightarrow{BC}}$

(d) $\underline{r} = (r\cos\theta, r\sin\theta, r\sqrt{3})$ $\quad 0 \le \theta < 2\pi$, $\quad 0 \le r < \infty$



* note this is a cone... check: if $\theta = 0 \Rightarrow \underbrace{\underline{r} = (r, 0, \sqrt{3} \cdot r)}$

This is a line "$y = \sqrt{3} \cdot x$"



Note
$$\frac{\partial \underline{r}}{\partial r} \times \frac{\partial \underline{r}}{\partial \theta} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & \sqrt{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= r(-\sqrt{3}\cos\theta, -\sqrt{3}\sin\theta, 1)$$

so normal is $\hat{\underline{n}} = \frac{1}{2}(\sqrt{3}\cos\theta, \sqrt{3}\sin\theta, -1)$

Remember to negate $\frac{\partial \underline{r}}{\partial r} \times \frac{\partial \underline{r}}{\partial \theta}$ to get normal pointing <u>down</u>.

#2. If $z = f(x,y)$, show $dS = \sqrt{1 + f_x^2 + f_y^2} \cdot dx \cdot dy$.

<u>Pf</u>: Use $\underline{r}(x,y) = (x, y, f(x,y))$

Compute $dS = \left| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right| \cdot dx \cdot dy$.

$$= \left| \begin{matrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{matrix} \right| \cdot dx \cdot dy$$

$|(-f_x, -f_y, 1)|$

$$= \sqrt{1 + f_x^2 + f_y^2} \cdot dx \cdot dy.$$

Show also $dS = \dfrac{dx\, dy}{|\hat{\underline{n}} \cdot \underline{k}|}$

<u>Pf</u>: The normal vector, as computed above,

is $\hat{\underline{n}} = \dfrac{(\underline{r}_x \times \underline{r}_y)}{|\underline{r}_x \times \underline{r}_y|} = \dfrac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}$

Thus $\hat{\underline{n}} \cdot \underline{k} = \dfrac{1}{\sqrt{1 + f_x^2 + f_y^2}}$ and we get

our result.

#3. a)

$$\underline{r} = (a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi)$$

$$(\underline{r}_\theta \times \underline{r}_\phi) = \begin{vmatrix} i & j & k \\ -a\sin\theta\sin\phi & a\cos\theta\sin\phi & 0 \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \end{vmatrix}$$

$$= -a^2(\cos\theta\sin^2\phi,\ \sin\theta\sin^2\phi,\ \sin\phi\cos\phi)$$

using identities $\cos^2\theta + \sin^2\theta = 1$.

$\Rightarrow |\underline{r}_\theta \times \underline{r}_\phi| = a^2\sin\phi$ after algebra.

$\therefore\ dS = a^2\sin\phi\ d\theta \cdot d\phi$

b) Find dS for face of tetrahedron

Since $dS = \dfrac{dx \cdot dy}{|\hat{\underline{n}} \cdot \underline{k}|}$ by #2

we only need $\hat{\underline{n}}$. We found $\hat{\underline{n}} = \dfrac{(bc, ac, ab)}{\sqrt{(bc)^2 + (ac)^2 + (ab)^2}}$

Thus $dS = \dfrac{dx \cdot dy}{\dfrac{ab}{\sqrt{(ab)^2 + (bc)^2 + (ac)^2}}}$

$$dS = \left\{\frac{\sqrt{(ab)^2 + (bc)^2 + (ac)^2}}{ab}\right\} \cdot dx \cdot dy$$

#3.(c) dS for the cone.

We calculated $\left(\frac{\partial \underline{r}}{\partial r} \times \frac{\partial \underline{r}}{\partial \theta}\right) = r\left(-\sqrt{3}\cos\theta, -\sqrt{3}\sin\theta, 1\right)$

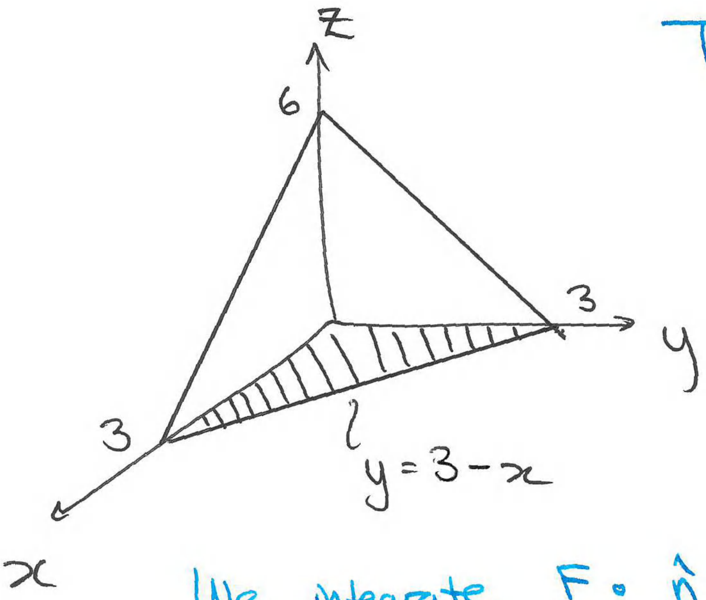
so $\left|\frac{\partial \underline{r}}{\partial r} \times \frac{\partial \underline{r}}{\partial \theta}\right| = 2r.$

Hence $dS = 2r \cdot dr \cdot d\theta$

#4. Since $a=3, b=3, c=6$,

$$\hat{\underline{n}} = \frac{(bc, ac, ab)}{\sqrt{\cdots}} = \frac{(18, 18, 9)}{|\cdots|} = \frac{1}{3}(2,2,1)$$

Thus $dS = \dfrac{dx \cdot dy}{|\hat{\underline{n}} \cdot \underline{k}|} = 3 \cdot dx \cdot dy.$



Thus,

$$d\underline{S} = \hat{\underline{n}}\, dS$$
$$= \frac{1}{3}(2,2,1)\left(3 \cdot dx \cdot dy\right)$$
$$= (2,2,1) \cdot dx \cdot dy.$$

We integrate $\underline{F} \cdot \hat{\underline{n}}\, dS$

$$= (z, 2xz, -2y) \cdot (2,2,1)\, dx \cdot dy$$

We need to know what is z on the planar region. Use the eqn of a plane:

$$\underline{x} \cdot \hat{\underline{n}} = \text{constant}$$

$$\Rightarrow \quad \underline{x} \cdot \frac{1}{3}(2,2,1) = c$$

Use the fact $(0,0,6)$ is a point:

$$(0,0,6) \cdot \frac{1}{3}(2,2,1) = 2 = c$$

$$\therefore \quad (x,y,z) \cdot \frac{1}{3}(2,2,1) = 2$$

$$\Rightarrow \quad 2x + 2y + z = 6 \Rightarrow z = 6 - 2x - 2y.$$

Thus $\underline{F} = (6-2x-2y,\ 2x(6-2x-2y),\ -2y)$

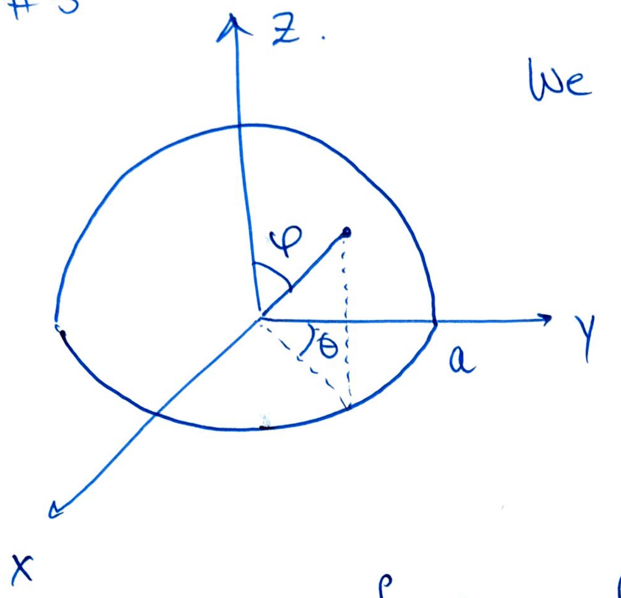
and $\underline{F} \cdot \hat{\underline{n}}\, dS = 2\{6 + 10x - 4x^2 - 3y - 4xy\}.$

$$\therefore \int_S \underline{F} \cdot \hat{n}\, dS = \iint_D 2\{6 + 10x - 4x^2 - 3y - 4xy\} \cdot dx \cdot dy$$

$$= \int_{x=0}^{3} \int_{y=0}^{3-x} [\ldots] \cdot dx \cdot dy = \underline{\underline{36}}$$

after computation!

#5



We shall use spherical coords

$$x = a\cos\theta\sin\varphi \qquad \theta \in [0, 2\pi)$$
$$y = a\sin\theta\sin\varphi \qquad \varphi \in [0, \tfrac{\pi}{2})$$
$$z = a\cos\varphi$$

$$I = \int_S d\underline{S} = \int_S \underline{n}\, dS = ?$$

Use $\underline{r}(\theta,\varphi)$ as above. Then $d\underline{S} = \underline{n}\, dS$

$$= (r_\theta \times r_\varphi)\, d\theta\, d\varphi$$

$$r_\theta \times r_\varphi = \begin{vmatrix} i & j & k \\ -a\sin\theta\sin\varphi & a\cos\theta\sin\varphi & 0 \\ a\cos\theta\cos\varphi & a\sin\theta\cos\varphi & -a\sin\varphi \end{vmatrix} = \underline{i}\,(-a^2\cos\theta\sin^2\varphi) - \underline{j}\,(a^2\sin\theta\sin^2\varphi) + \underline{k}\,(-a^2\sin^2\theta\sin\varphi\cos\varphi - a^2\cos^2\theta\sin\varphi\cos\varphi)$$

$$= \underbrace{-a^2\sin\varphi\,[\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi]}$$

note you could predict this! It's $-J \cdot \dfrac{[x, y, z]}{\sqrt{x^2+y^2+z^2}}$

where $J = a^2\sin\varphi$.

$$\therefore \quad I = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\frac{\pi}{2}} a^2 \sin\varphi \,[\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi] \cdot d\varphi \cdot d\theta$$

don't forget to negate the cross-product to get outer normal.

now $\displaystyle\int_0^{2\pi} \cos\theta \cdot d\theta = 0 = \int_0^{2\pi} \sin\theta \cdot d\theta$

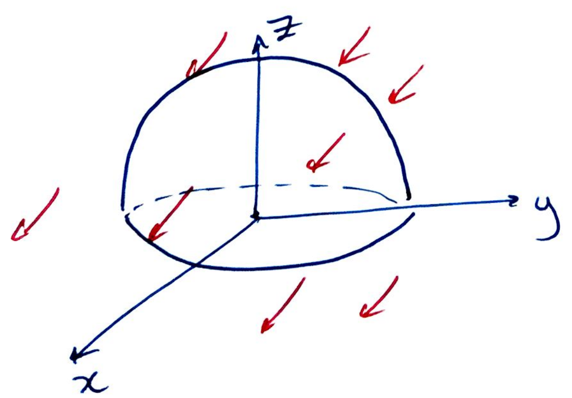
Thus only $\underline{k}$ component remains:

$$I = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\frac{\pi}{2}} \underline{k}\, a^2 \sin\varphi \cos\varphi \cdot d\varphi \cdot d\theta$$

$$= \underline{k}\, a^2 (2\pi) \int_{\varphi=0}^{\frac{\pi}{2}} \left(\tfrac{1}{2}\sin 2\varphi\right) \cdot d\varphi = \underline{k}\, a^2 \pi$$

(b) Consider $\displaystyle\int_S \underline{F} \circ \hat{\underline{n}}\, dS$ where $\underline{F} = \hat{e}_i \quad (\underline{i}, \underline{j}, \underline{k})$

Now $\displaystyle\int_S \underline{e}_1 \circ \hat{\underline{n}}\, dS = \int_S \underline{i} \cdot \hat{\underline{n}}\, dS$ = Flux due to a field that points in x-direction



But we expect

$$\int_{S,\ x \geq 0} \underline{F} \circ \hat{\underline{n}}\, dS = - \int_{S,\ x \leq 0} \underline{F} \circ \hat{\underline{n}}\, dS$$

since the flow into the "left upper hemisphere" should cancel the flow into the "right upper hemisphere".

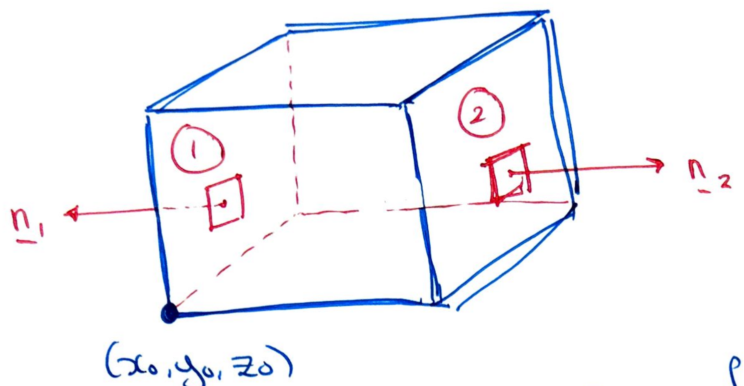
Similarly, you expect if $\underline{F} = \underline{\hat{e}}_2 = \hat{j}$ that

$$\int_{\substack{S \\ y \leq 0}} \underline{F} \cdot \underline{\hat{n}} \, dS + \int_{\substack{S \\ y \geq 0}} \underline{F} \cdot \underline{\hat{n}} \, dS = 0 .$$

So the only case that yields something non-zero is

$$\int_S \underline{F} \cdot \underline{\hat{n}} \, dS = \int_S \underline{\hat{e}_3} \cdot \underline{\hat{n}} \, dS .$$

#6.



We wish to calculate $Q = \iint_S \underline{F} \cdot \hat{\underline{n}}\, dS$ for a cube centred at (x_0, y_0, z_0).

Consider firstly $\left(\iint_{S_1} + \iint_{S_2} \right) \underline{F} \cdot \hat{\underline{n}}\, dS$. Then:

S_1: $\underline{n}_1 = (-1, 0, 0)$ $\quad \underline{F} \cdot \underline{n}_1 = -F_1(x, y, z)$

S_2: $\underline{n}_2 = (+1, 0, 0)$ $\quad \underline{F} \cdot \underline{n}_2 = F_1(x, y, z)$.

Now on S_1, F can be expanded as follows:

$$F_1(x, y, z) \approx F_1(x_0, y_0, z_0)$$

while on S_2,

$$F_1(x, y, z) \approx F_1(x_0, y_0, z_0) + \frac{\partial F_1}{\partial x}(x_0, y_0, z_0) \cdot (\delta x) + \ldots$$

by Taylor series.

Then $\left(\iint\limits_{S_1} + \iint\limits_{S_2}\right) \underline{F} \cdot \underline{\hat{n}} \, dS \approx -F_1(x_0, y_0, z_0) \cdot \iint\limits_{S_1} dS$

$$+ \left(F_1(x_0, y_0, z_0) + \frac{\partial F_1}{\partial x}(x_0, y_0, z_0)\,\delta x + \dots\right) \iint\limits_{S_2} dS$$

$$= \left[F_1(x_0, y_0, z_0) - F_1(x_0, y_0, z_0)\right] \cdot \delta y \cdot \delta z$$

$$+ \frac{\partial F_1}{\partial x}(x_0, y_0, z_0)\,\delta x \cdot \delta y\,\delta z$$

$$= \frac{\partial F_1}{\partial x}(x_0, y_0, z_0)\,\delta x \cdot \delta y\,\delta z\,.$$

Thus by symmetry

$$Q \approx \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \delta x \cdot \delta y\,\delta z\,.$$