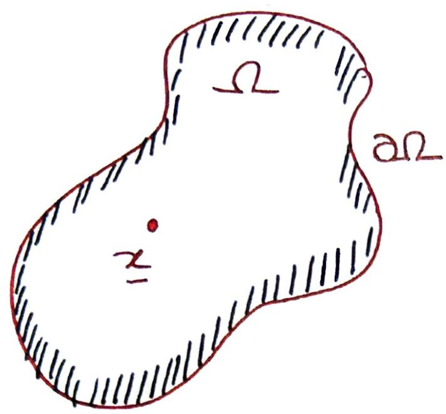
Problem Set # 5

Q1 :



This is a standard derivation of the heat eqn for a homogeneous, continuous, isotropic solid.

By energy conservation applied to Ω:

$$\underbrace{\frac{d}{dt}\iiint_{\Omega} \rho c T \cdot dV}_{①} = \underbrace{-\iint_{\partial\Omega} \underline{q} \cdot \underline{\hat{n}} \, dS}_{②}$$

The LHS ① is the change in total internal heat since the heat energy at a point is given by

$$E = \rho c T \leftarrow \text{temperature.}$$

density (ρ), specific heat or heat per unit mass (c)

The RHS ② is the amount of heat leaving the boundary, $\partial\Omega$, where $\underline{q}$ is the flux vector. Thus $(\underline{q} \cdot \underline{\hat{n}})\, dS$ gives the heat leaving dS.

* note $(\underline{q} \circ \underline{\hat{n}} \cdot dS) > 0$ means heat is leaving
so $\iiint \rho c T \, dV$ should decrease, hence the
negative sign.

By Fourier's Law: $\underline{q} = -k \nabla T$

$$\therefore \iiint_{\Omega} \frac{\partial T}{\partial t} \cdot dV = + \iint_{\partial\Omega} k \nabla T \cdot \underline{\hat{n}} \; dS$$

By the divergence thm,

$$k \iint_{\partial\Omega} (\nabla T) \cdot \underline{\hat{n}} \; dS = k \iiint_{\partial\Omega} \nabla \cdot (\nabla T) \; dV$$

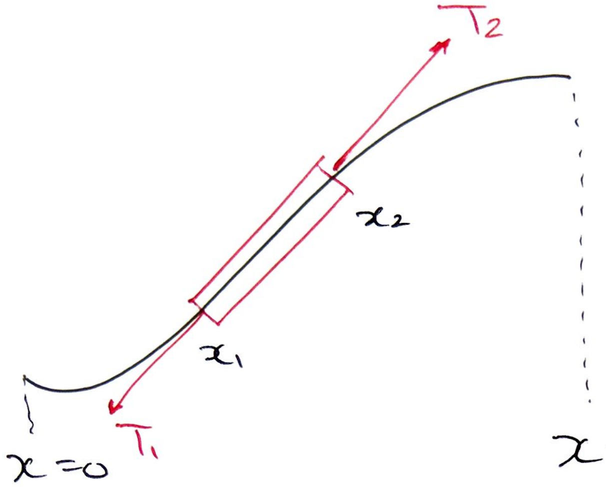
$$= k \cdot \iiint_{\Omega} \nabla^2 T \cdot dV$$

Since this is true for the integral*, we claim
it is true for the integrand

$$\frac{\partial T}{\partial t} = k \nabla^2 T.$$

* The reason why is that we could have used any
subset $\Omega^* \subseteq \Omega$ and the demonstration would
still be true.

DERIVATION OF WAVE EQUATION.

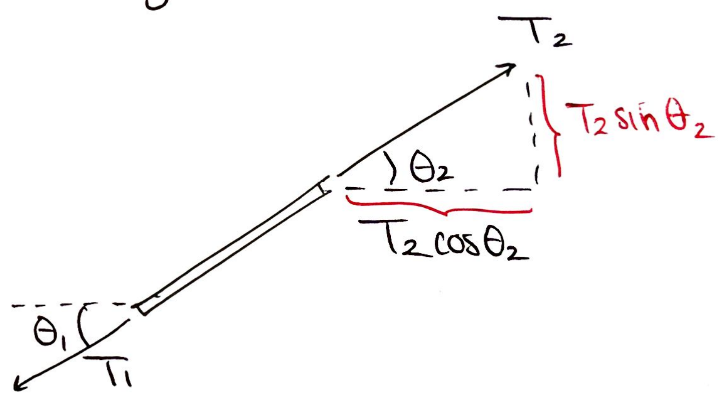


Assumptions:

- String tension is T
- String density is ρ
- Gravity and air resistance ignored.

- We also assume small deflection $\Rightarrow \left|\frac{\partial y}{\partial x}\right|$ is small.

- Consider Newton's Laws on a segment $[x_1, x_2]$



Horizontal forces balance like

$$T_2 \cos\theta_2 = T_1 \cos\theta_1 \qquad (1)$$

Vertical forces balance like

$$(\rho \Delta x) \frac{\partial^2 y}{\partial t^2}(x_0, t) = T_2 \sin\theta_2 - T_1 \sin\theta_1 \quad (2) \text{ where } x_0 \in [x_1, x_2].$$

we should use $\Delta S = \sqrt{\Delta x^2 + \Delta y^2} = \Delta x \sqrt{1 + (\Delta y/\Delta x)^2} \sim \Delta x$ since $\Delta y/\Delta x$ small.

- Since $|y_x|$ small $T_2 \cos\theta_2 \sim T_2$ and $T_1 \cos\theta_1 \sim T_1$ since θ_2 and θ_1 are small. From (1)

$$\therefore T_2 \sim T_1$$

- Now $\sin\theta \sim \tan\theta$ if θ is small (since $\cos\theta \sim 1$)

$$\sim \frac{\partial y}{\partial x}$$

$$\therefore \sin\theta_1 \sim \frac{\partial y}{\partial x}(x_1, t) \quad \text{and} \quad \sin\theta_2 \sim \frac{\partial y}{\partial x}(x_2, t)$$

Thus (2) $\Rightarrow$ $\rho \Delta x \frac{\partial^2 y}{\partial t^2}(x_0, t) \sim T \frac{\partial y}{\partial x}(x_2, t) - \frac{\partial y}{\partial x}(x_1, t)$

use $x_2 = x_1 + \Delta x$ $\Rightarrow$ $y_{tt}(x_0, t) \sim \frac{T}{\rho} \frac{\partial^2 y}{\partial x^2}(a, t)$

where $a \in [x_1, x_2]$ by the MVT. Thus

$$y_{tt}(x, t) = \frac{T}{\rho} y_{xx}(x, t)$$

once we take $\Delta x \to 0$.

#3. (a) $u(0,t) = T_0$, $u(L,t) = T_1$ means fix the temperature at $x = 0, L$.

$\frac{\partial u}{\partial x}(0,t) = F_0$, $\frac{\partial u}{\partial x}(L,t) = F_1$ means fix the heat flux @ $x = 0, L$

(Recall $q = -k\frac{\partial T}{\partial x}$ so we are fixing the rate that heat is pumped in/out)

(b) Thermal insulation means that we do not allow heat flow $\Rightarrow q = -k\frac{\partial T}{\partial x} = 0$.

* note this is not the same as saying the boundary is fixed at a temperature.

(c) Fixed temperature $\Rightarrow T(\underline{x},t) = G(\underline{x},t)$ for all $\underline{x} \in S$. In the case the temperature is fixed for all time, $T(\underline{x},t) = G(\underline{x})$ for $\underline{x} \in S$.

Fixed flux $\Rightarrow \underline{q}\cdot\hat{\underline{n}} = H(\underline{x},t)$ or $\nabla T\cdot\underline{n} = H(\underline{x},t)$

Thermal insulation $\Rightarrow \nabla T\cdot\underline{n} = 0$.

Q4:

$$\begin{cases} y_{tt} = c^2 y_{xx} \\ y(0,t) = 0 = y(L,t), \quad t > 0. \end{cases}$$

Assume $y(x,t) = X(x) \cdot T(t)$

$$\Rightarrow T''X = c^2 X''T$$

$$\Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X}$$

Since LHS is only dep. on time and RHS only dependent on x, we must have

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda^2 < 0 \quad (+)$$

The constant chosen to obtain non-trivial sol'ns.

Then
$$X = A\cos(\lambda t) + B\sin(\lambda t)$$
$$T = C\cos(\lambda c t) + D\sin(\lambda c t).$$

BCs:
$$y(0,t) = 0 \Rightarrow X(0)\,T(t) = 0 \Rightarrow X(0) = 0$$
$$y(L,t) = 0 \Rightarrow X(L)\,T(t) = 0 \Rightarrow X(L) = 0$$

So we must have $A = 0$ and

$$B\sin(\lambda L) = 0 \Rightarrow \lambda L = n\pi, \quad n \in \mathbb{Z}$$

For different n we should choose

$$y_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right)\left\{C_n \cos\left(\frac{n\pi c t}{L}\right) + D_n \sin\left(\frac{n\pi c t}{L}\right)\right\}$$

The n should be $-3, -2, -1, 0, 1, 2, 3, \dots \in \mathbb{Z}$.

But negative n can be absorbed into the coefs while $n=0$ is $y_n = 0$.

What happens if $\lambda^2 = 0$ in (+) ?

Then $T'' = 0 \Rightarrow T = At + B$. To keep solutions from growing $t \to \infty$ need $A = 0$.

Similarly $X'' = 0 \Rightarrow X = Cx + D$.

But $X(0) = 0 = X(L) \Rightarrow C = D = 0$.

Can verify trivial solutions for "$-\lambda^2$" > 0. So we conclude

$$y = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left\{\hat{C}_n \cos\left(\frac{n\pi c t}{L}\right) + \hat{D}_n \sin\left(\frac{n\pi c t}{L}\right)\right\}$$

where $\hat{C}_n = B_n C_n$, $\hat{D}_n = B_n D_n$.

next chapter we try to understand these functions.