1.

$$x_{n+1} = f(x_n, \mu) = \mu x_n(1-x_n)$$

a) <u>fixed points</u>

$$x = \mu x(1-x)$$

$\Rightarrow x = 0$ (exists for all μ)

or

$x^* = 1 - \frac{1}{\mu}$ (exists for $\mu \neq 0$)

Non-trivial fixed point is in $[0,1]$ if $\mu > 1$.

b) <u>Period doubling</u>

$$f'(x) = \mu(1-2x).$$

So $f'(0) = \mu$

and $f'(1-\frac{1}{\mu}) = \mu(-1+\frac{2}{\mu}) = 2-\mu.$

Hence $f'(1-\frac{1}{\mu}) = -1$ when $\mu = 3$.

c) <u>2-cycle</u>

Solve $f^2(x) = x$ to find the 2-cycle.

$$f^2(x) = x \Rightarrow \mu[\mu x(1-x)][1-\mu x(1-x)] = x$$

$$\Rightarrow x(x-1+\tfrac{1}{\mu})(x^2 - x(1+\tfrac{1}{\mu}) + \tfrac{1}{\mu}(1+\tfrac{1}{\mu})) = 0$$

The first two factors are the fixed points of f found in (a).

The quadratic factor gives

$$x_{1,2} = \frac{1}{2}\left(1+\frac{1}{\mu}\right) \pm \frac{1}{2}\left[\left(1+\frac{1}{\mu}\right)\left(1-\frac{3}{\mu}\right)\right]^{1/2}$$

$x_{1,2}$ are real if and only if $\mu \geq 3$.

Hence 2-cycle exists for $\mu > 3$, but not $\mu < 3$.
At $\mu = 3$ $\quad x_1 = x_2 = \frac{1}{2}\left(1+\frac{1}{3}\right) = \frac{2}{3} = 1 - \frac{1}{3} = 1 - \frac{1}{\mu} = x^*$.

d)

Let $G(x) := f^2(x)$.

Chain rule: $\quad G'(x) = f'(f(x)).f'(x)$.

So, on the 2-cycle $\{x_1, x_2\}$,

$$G'(x_1) = f'(x_2)f'(x_1) = f'(x_1)f'(x_2) = G'(x_2).$$

Hence G' is constant on $\{x_1, x_2\}$.

<u>Stability</u>

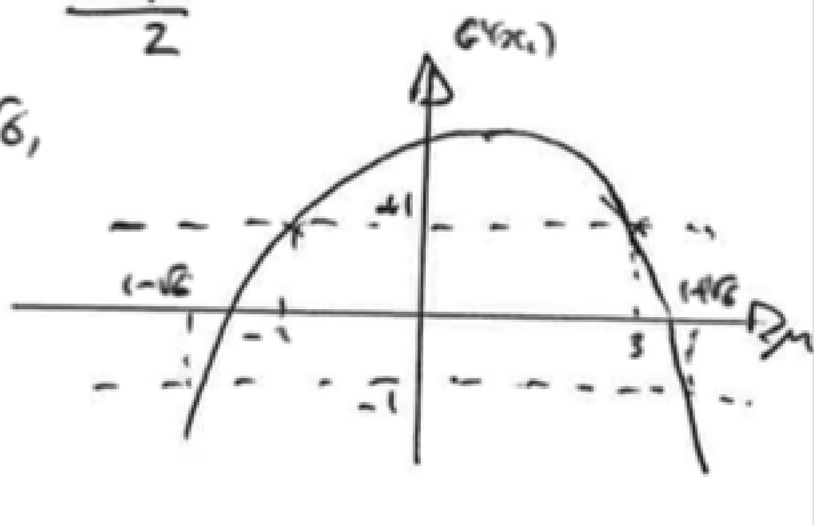
x_1 and x_2 fixed points of G.
Perturbations near 2-cycle decay if $|G'(x)| < 1$.

$$G'(x_1) = \mu^2(1-2x_1)(1-2x_2)$$
$$= 4 + 2\mu - \mu^2.$$

So $\quad G'(x_1) = 1 \quad$ when $\mu^2 - 2\mu - 3 = 0$
$\Rightarrow (\mu-3)(\mu+1) = 0$
$\Rightarrow \mu = -1, 3$.

$G'(x_1) = -1 \quad$ when $\mu^2 - 2\mu - 5 = 0$
$\Rightarrow \mu = \frac{2 \pm \sqrt{4+20}}{2} = 1 \pm \sqrt{6}$

Hence $|G'(x_1)| < 1$ if $3 \leq \mu < 1+\sqrt{6}$,
and the 2-cycle is stable
in this range.



e) At $\mu = 1+\sqrt{6}$ there is a period-doubling of the 2-cycle. This generates a 4-cycle.