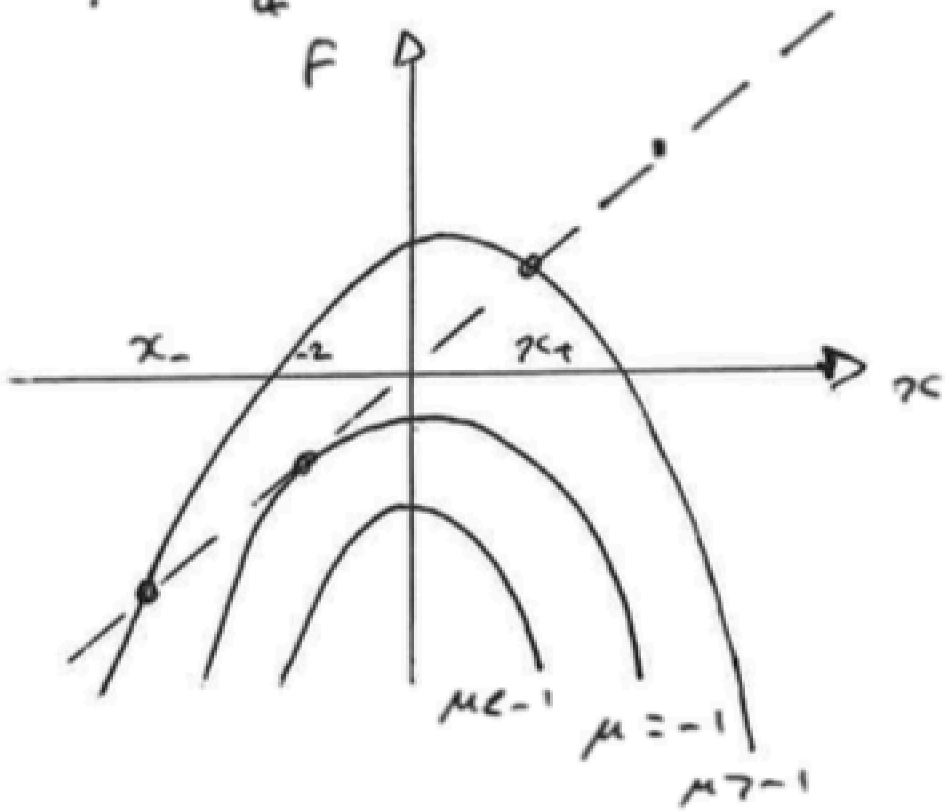
2. $$x_{n+1} = \mu - \frac{x_n^2}{4} =: F(x_n,\mu)$$



Fixed points

$$x = \mu - \frac{x^2}{4}$$

$$\Rightarrow x_\pm = \frac{-1 \pm [1+\mu]^{1/2}}{1/2} = -2(1 \mp [1+\mu]^{1/2}), \quad (\mu \geq -1)$$

Stability

$$F_x(x,\mu) = -\frac{x}{2}.$$

So $F_x(x_+) = 1 - [1+\mu]^{1/2}$
and $|F_x(x_+)| < 1$ if $-1 < \mu < 3$.

Also $F_x(x_-) = 1 + [1+\mu]^{1/2}$
and $|F_x(x_-)| > 1$, for all μ.

Hence: no fixed points for $\mu < -1$.
at $\mu = -1$, saddle-node bifurcation – pair of stable-unstable fixed points from $x = -2$
at $\mu = 3$, $f_x(x_+, \mu) = -1$, period-doubling bifurcation from $x = 2$

Period-2 points

$$f^2(x) = \mu - \frac{1}{4}\left[\mu - \frac{x^2}{4}\right]^2$$

$$= \mu - \frac{\mu^2}{4} + \frac{\mu x^2}{8} - \frac{x^4}{64}$$

So $x = f^2(x)$
if $x^4 - 8\mu x^2 + 64x + 16\mu^2 - 64\mu = 0$
$\Rightarrow \underbrace{(x^2 + 4x - 4\mu)}_{\text{fixed points of } f}\underbrace{(x^2 - 4x - 4\mu + 16)}_{\text{Additional fixed points of } f^2} = 0$

$\Rightarrow \quad x_{1,2} = 2 \pm 2[\mu - 3]^{1/2}$ are the period-2 points.

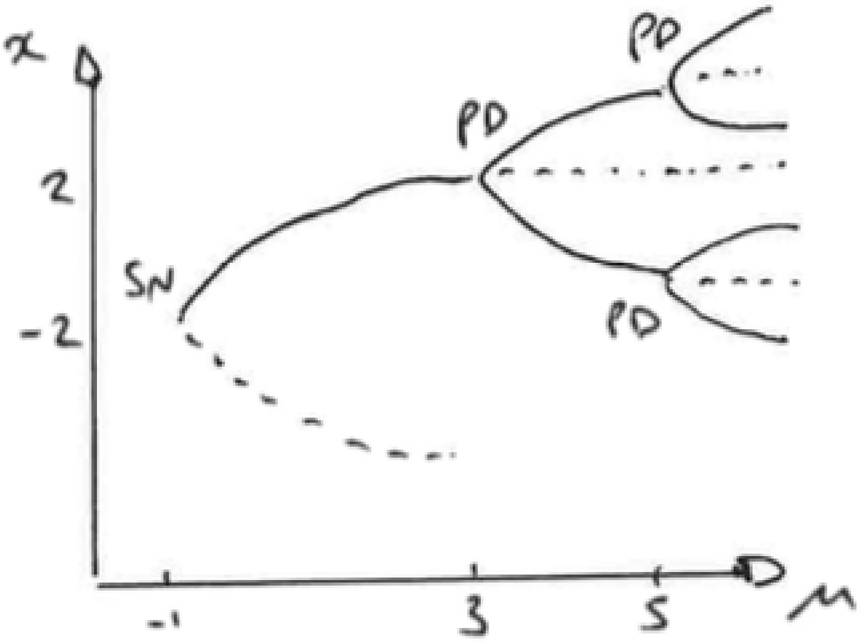
Stability

$$f_x(x_1) f_x(x_2) = \frac{d}{dx} f^2(x)\Big|_{x = x_1 \text{ or } x_2}$$

$$= \frac{1}{4} x_1 x_2$$

$$= 1 - (\mu - 3) = 4 - \mu.$$

Hence period-2 orbit stable for $3 < \mu < 5$, and has a period-doubling bifurcation at $\mu = 5$.



x
2
-2
SN
PD
PD
PD
-1
3
5
μ