4.

$$x_{n+1} = F(x_n) = 4x_n(1-x_n) \quad \text{on} \quad X = [0,1]$$

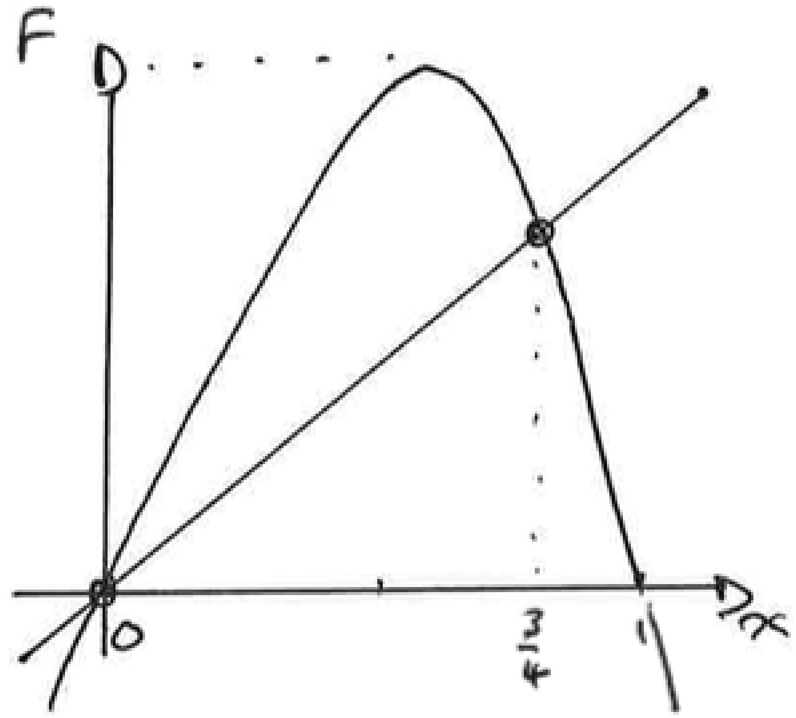
$$y_{n+1} = G(y_n) = 2 - y_n^2 \quad \text{on} \quad Y \subseteq \mathbb{R}$$

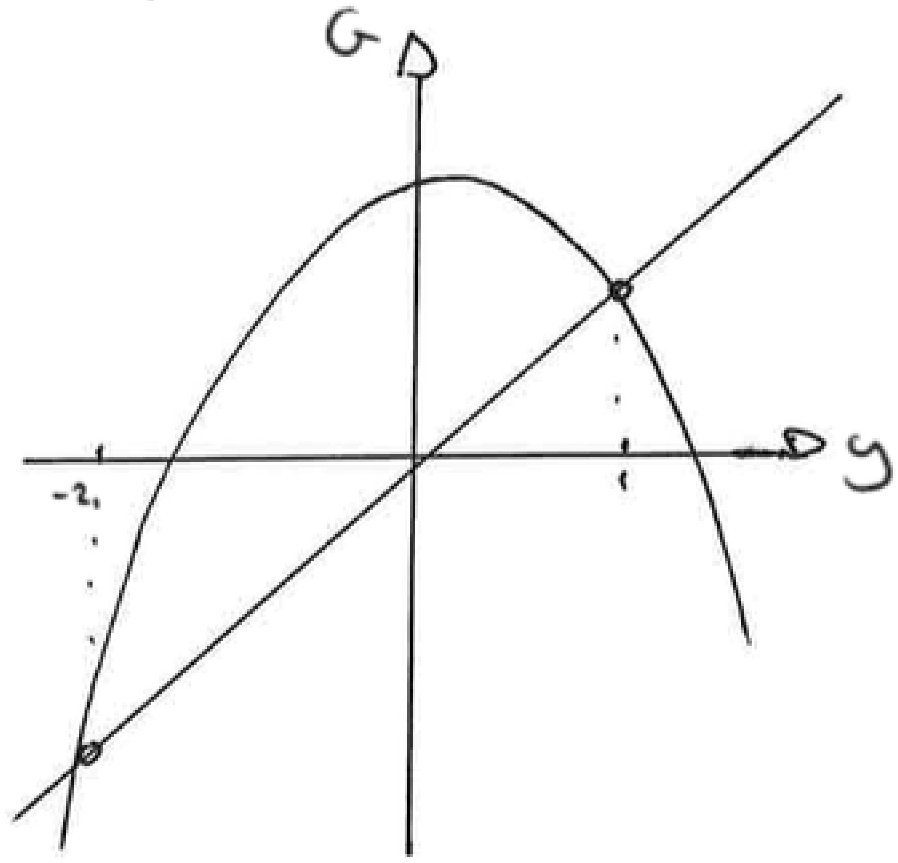
a) F has fixed points ~~a~~ when $4x(1-x) = x$

$\Rightarrow \quad 3x - 4x^2 = 0$

$\Rightarrow \quad x = 0 \text{ or } \frac{3}{4}$

G has fixed points when $2-y^2=y$
$\Rightarrow y^2+y-2=0$
$\Rightarrow y=1$ or -2.





Find homeomorphism h
Require $h\circ f(x) = G\circ h(x)$, and fixed points must map to fixed points.

Try $h(x) = ax+b$ for a, b to be found.

If h maps fixed points to fixed points

$$\left.\begin{aligned} h(0) &= -2 \\ h(3/4) &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} b &= -2 \\ a &= 4. \end{aligned}$$

So $h(x) = 4x-2$ could work, if it is a homeomorphism.

Now $h(x)$ is clearly continuous and invertible

$h(X) = h[0,1] = [-2,2] = Y.$

Also

$$
\begin{aligned}
h \circ f(x) &= h(4x(1-x)) \\
&= 16x(1-x) - 2 \\
&= -2 + 16x - 16x^2 \\
&= 2 - (4 - 16x + 16x^2) = 2 - (4x-2)^2 \\
&= 2 - (h(x))^2 \\
&= G \circ h(x)
\end{aligned}
$$

Hence $h(x) = 4x-2$ provides a conjugacy between all orbits of f and G, not just the fixed points.

b)

$$f(x,\mu) = \mu x(1-x) \quad \text{on } X$$
$$G(y,\lambda) = \lambda - y^2. \quad \text{on } Y.$$

Let h be a homeomorphism between F and G.

Try $h(x) = ax+b.$

Then $h \circ F(x) = G \circ h(x)$ requires

$$h(\mu x(1-x)) = \lambda - h(x)^2$$
$$\Rightarrow \quad a\mu x(1-x) + b = \lambda - a^2x^2 - 2abx - b^2$$
$$\Rightarrow \quad a = \mu, \quad b = -\frac{\mu}{2}, \quad \lambda = b^2 + b$$

which means that $\lambda = \frac{\mu}{2}\left(\frac{\mu}{2} - 1\right).$

or equivalently $\mu = 1 \pm [1+4\lambda]^{1/2}$

Hence require $\lambda \geq -1/4$, and to ensure unique μ for any given λ, require $\mu > 1$.