1. a) $$x_{n+1} = 4\mu - (\mu+3)x_n + x_n^2$$

Fixed points

$$x = 4\mu - (\mu+3)x + x^2$$

$$\Rightarrow \; 0 = x^2 - (\mu+4)x + 4\mu$$

$$\Rightarrow \; 0 = (x-\mu)(x-4)$$

$$\Rightarrow \; x = \cancel{3}4 \text{ or } x = \mu.$$

Both of these fixed points exist for all μ.

Stability

$$f_x(x,\mu) = -(\mu+3) + 2x$$

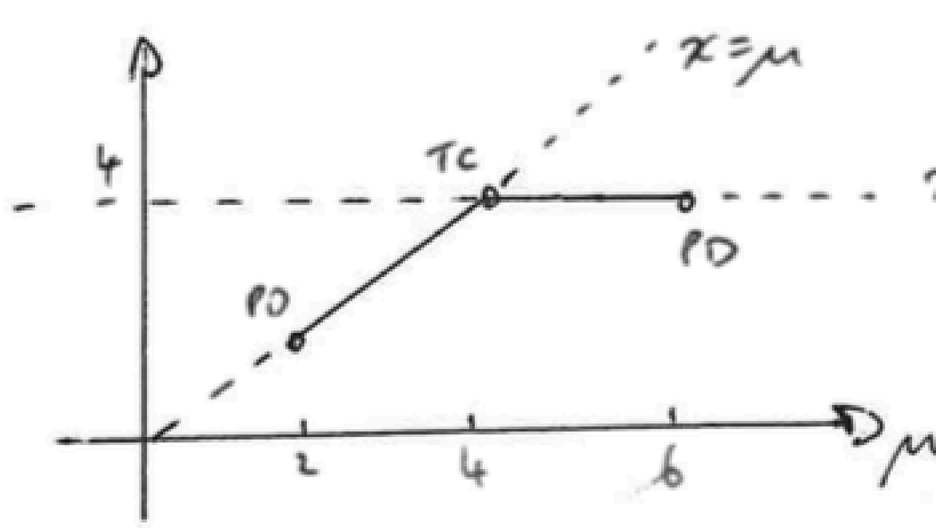
So $f_x(4,\mu) = 5-\mu$ and $x=4$ is stable for $4<\mu<6$.

$f_x(\mu,\mu) = \mu - 3$ and $x=\mu$ is stable for $2<\mu<4$.

At $\mu=2$, $f_x(\mu,\mu) = -1$, so looks like period-doubling ~~fixed~~ bifurcation at fixed point $x=\mu$.

At $\mu=4$, $f_x(4,\mu) = +1$ and $f_x(\mu,\mu) = +1$. Both fixed points exist either side of $\mu=4$, so looks like transcritical bifurcation.

At $\mu=6$, $f_x(4,\mu) = -1$, looks like period-doubling bifurcation at fixed point $x=4$.



Near $\mu = 4$

Let $y = x - 4$, $\lambda = \mu - 4$.

Write $x_{n+1} = 4\mu - (\mu+3)x_n + x_n^2$

as $x_{n+1} = (4 - x_n)(\mu - x_n) + x_n$

Then
$$\begin{aligned} y_{n+1} &= x_{n+1} - 4 \\ &= -y_n(\lambda - y_n) - 4 + x_n \\ &= -\lambda y_n + y_n^2 + y_n \\ &= (1-\lambda)y_n + y_n^2. \\ &= y_n - \lambda y_n + \lambda_n^2 \end{aligned}$$

So, with refeence to the bifurcation classification table, $a_0 = 1$, $a_1 = 0$, $b_0 = 1$, $b_1 = -1$ which indicates a transcritical bifurcation.

Near $\mu = 2$

Let $y = x - 2 - \lambda$, $\lambda = \mu - 2$.

So $\mu - x = -y$ and $4 - x = 2 - y + \lambda$.

Then
$$\begin{aligned} y_{n+1} &= x_{n+1} - 2 - \lambda \\ &= (4 - x_n)(\mu - x_n) + x_n - 2 - \lambda \\ &= (2 - \lambda - y_n)(-y_n) + y_n \\ &= (\lambda - 1)y_n + y_n^2 \\ &= -y_n + \lambda y_n + y_n^2 \end{aligned}$$

So $a_0 = -1$, $a_1 = 0$, $b_0 = 1$, $b_1 = 1$, which indicates a period-doubling bifurcation,

Consider second iterate to determine criticality:

$$\begin{aligned} y_{n+2} &= \cancel{(-1\,2\lambda}\,(\lambda - 1)[(\lambda-1)y_n + y_n^2] + [(\lambda-1)y_n + y_n^2]^2 \\ &= (\lambda^2 - 2\lambda + 1)y_n + (\lambda-1)y_n^2 + (\lambda^2 - 2\lambda + 1)y_n^2 \\ &\quad + y_n^4 + 2(\lambda-1)y_n^3 \end{aligned}$$

$$= (1-2\lambda)y_n + (\lambda^2-\lambda)y_n^2 - 2y_n^3 + \lambda^2 y_n + O(|\lambda, y_n|^4)$$
$$= (1-2\lambda)y_n - 2y_n^3 - \lambda y_n^2 + \lambda^2 y_n + O(|\lambda, y_n|^4).$$

So, 2-cycle exists ($y_{n+2} = y_n$) if $y \sim O(\lambda^{1/2})$ and $y \approx (1-2\lambda)y - 2y^3$.

$\Rightarrow \lambda \approx -y^2 < 0$

$\Rightarrow \mu < 2.$

If $\mu < 2$, the fixed point at $x=\mu$ is unstable. Hence the 2-cycle is stable and the period-doubling bifurcation is supercritical.

<u>Near $\mu = 6$</u>

Let $y = x-4$, $\lambda = \mu - 6$

Then
$$\begin{aligned} y_{n+1} &= x_{n+1} - 4 \\ &= (4-x_n)(\mu - x_n) + x_n - 4 \\ &= (-y_n)(\lambda + 2 - y_n) + y_n \\ &= -(1+\lambda)y_n + y_n^2 \\ &= -y_n - \lambda y_n + y_n^2 \end{aligned}$$

So $a_0 = -1$, $a_1 = 0$, $b_0 = -1$, $b_1 = 1$ which indicates a period doubling bifurcation.

The second iterate is

$$y_{n+2} = (1+2\lambda)y_n - 2y_n^3 + \lambda y_n^2 + \lambda^2 y_n + O(|y_n, \lambda|^4)$$

So 2-cycle exists if $y \sim O(\lambda^{1/2})$ and $y \approx (1+2\lambda)y - 2y^3 \Rightarrow \lambda \approx y^2 > 0.$

$\Rightarrow \mu > 6.$

If $\mu > 6$ then fixed point $x = 4$ unstable. So period-doubling bifurcation supercritical.

b) $$x_{n+1} = \mu - 2 + \mu x_n + x_n + 3x_n^2 + x_n^3 =: f(x_n, \mu).$$

<u>fixed points</u>

$$x = f(x, \mu)$$
$$\Rightarrow \quad 0 = \mu - 2 + \mu x + 3x^2 + x^3$$
$$= (x+1)(x^2 + 2x + \mu - 2)$$
$$\Rightarrow \quad x = -1 \quad \text{or} \quad x_\pm = -1 \pm \sqrt{3-\mu} \quad (\text{exists if } \mu < 3).$$

<u>Stability</u>

$$f_x(x, \mu) = \mu + 1 + 6x + 3x^2$$

So $f_x(-1, \mu) = \mu - 2$, stable if $1 \leq \mu < 3$.

$$F_x(x_+, \mu) = \mu + 1 + 6[-1 + \sqrt{3-\mu}] + 3[-1 + \sqrt{3-\mu}]^2$$
$$= \mu + 1 - 6 + 6\sqrt{3-\mu} + 3[1 - 2\sqrt{3-\mu} + (3-\mu)]$$
$$= \mu - 2 + 9 - 3\mu$$
$$= 7 - 2\mu.$$

Similarly $f_x(x_-, \mu) = 7 - 2\mu$.

So $x_\pm$ unstable if they exist ($\mu < 3$).

<u>Bifurcation</u>

At $\mu = 3$, three fixed points become one (as μ increases). For $\mu < 3$ two of these fixed points are unstable.

So subcritical pitchfork bifurcation at $\mu = 3$

<u>Canonical form</u>

Let $y = x + 1$ and $\lambda = \mu - 3$.

Then $$y_{n+1} = x_{n+1} + 1$$
$$= (x_n + 1)(x_n^2 + 2x_n + \mu - 2) + x_n + 1$$

$$= y_n(y_n^2 + \lambda) + y_n$$
$$= (1+\lambda)y_n + y_n^3$$

This is the canonical form for a subcritical pitchfork bifurcation.