2. Let $f: I \to \mathbb{R}$ have a 4-cycle $\{x_0, x_1, x_2, x_3\}$.

i) a) Suppose $x_0 < x_1 < x_2 < x_3$.
Let $I_0 = [x_0, x_1]$, $I_1 = [x_1, x_2]$, $I_2 = [x_2, x_3]$.

Then
$$f(I_0) \supseteq [f(x_0), f(x_1)] = [x_1, x_2] = I_1$$
$$f(I_1) \supseteq [f(x_1), f(x_2)] = [x_2, x_3] = I_2$$
$$f(I_2) \supseteq [f(x_3), f(x_2)] = [x_0, x_3] = I_0 \cup I_1 \cup I_2.$$

Hence the transition graph Γ of f-covering relations is

$$\begin{tikzcd}
I_0 \arrow[rr] & & I_1 \arrow[dl, shift left] \\
& I_2 \arrow[ul] \arrow[ur, shift left] \arrow[loop below] &
\end{tikzcd}$$

This transition graph does not indicate a horseshoe.

However, the induced graph of f^2-covering relations is

$$\begin{tikzcd} I_0 & I_1 \arrow[l] \arrow[loop right] \arrow[d, shift left] \\ & I_2 \arrow[ul, shift left] \arrow[u, shift left] \arrow[loop below] \end{tikzcd}$$

$$\begin{tikzcd} I_0 \arrow[dr, shift left] & \\ & I_2 \arrow[ul, shift left] \end{tikzcd}$$

This graph has a subgraph (the I_1, I_2 section) ~~with~~ that indicates f^2 has a horseshoe.

Hence f is chaotic.

b) Suppose $x_1 < x_0 < x_2 < x_3$

Let $I_0 = [x_1, x_0]$, $I_1 = [x_0, x_2]$, $I_2 = [x_2, x_3]$.

Then
$$\begin{aligned} f(I_0) &\supseteq [x_1, x_2] = I_0 \cup I_1 \\ f(I_1) &\supseteq [x_1, x_3] = I_0 \cup I_1 \cup I_2 \\ f(I_2) &\supseteq [x_0, x_3] = I_1 \cup I_2 \end{aligned}$$

Hence the transition graph is

$$\begin{tikzcd} I_0 \arrow[loop left] \arrow[r, shift left] & I_1 \arrow[l, shift left] \arrow[loop right] \arrow[d, shift left] \\ & I_2 \arrow[u, shift left] \arrow[loop below] \end{tikzcd}$$

This graph indicates that f has two horseshoes, and so is chaotic.

c) Suppose $x_3 < x_1 < x_2 < x_0$

Let $I_0 = [x_3, x_1]$, $I_1 = [x_1, x_2]$, $I_2 = [x_2, x_0]$

Then
$$f(I_0) \supseteq [x_2, x_0] = I_2.$$
$$f(I_1) \supseteq [x_3, x_2] = I_0 \cup I_1$$
$$f(I_2) \supseteq [x_3, x_1] = I_0$$

Hence the transition graph is

$$\begin{tikzcd} I_0 \arrow[dr, shift left] & I_1 \arrow[l] \arrow[loop right] \\ & I_2 \arrow[ul, shift left] \end{tikzcd}$$

This graph does not show a horseshoe. Furthermore, $f^n(I_0)$ cannot cover for any n, so $\{I_0, I_1\}$ cannot form a horseshoe for any n. Furthermore, for any n, either $f^n(I_0) \to I_2$ and $f^n(I_2) \to I_0$ <u>or</u> $f^n(I_0) \to I_0$ and $f^n(I_2) \to I_2$, but not <u>both</u>. So $\{I_0, I_2\}$ cannot form a horseshoe for any n. Clearly $\{I_1, I_2\}$ cannot form a horseshoe for any n either.

Hence f is not chaotic.

d). Suppose $x_1 < x_2 < x_0 < x_3$

Let $I_0 = [x_1, x_2]$, $I_1 = [x_2, x_0]$, $I_2 = [x_0, x_3]$.

Then
$$f(I_0) \supseteq I_1 \cup I_2$$
$$f(I_1) \supseteq I_1 \cup I_2 \cup I_3$$
$$f(I_2) \supseteq I_0 \cup I_1.$$

Hence the transition graph is

$$\begin{tikzcd}
I_0 \arrow[rr, shift left] \arrow[dr, shift left] & & I_1 \arrow[ll, shift left] \arrow[dl, shift left] \arrow[loop right] \\
& I_2 \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

This graph does not show a horseshoe. However, the induced graph for f^2 is

$$\begin{tikzcd}
I_0 \arrow[loop left] \arrow[rr, shift left] \arrow[dr, shift left] & & I_1 \arrow[ll, shift left] \arrow[dl, shift left] \arrow[loop right] \\
& I_2 \arrow[ul, shift left] \arrow[ur, shift left] \arrow[loop below] &
\end{tikzcd}$$

which has horseshoes everywhere. Hence f is chaotic.