

3. Sawtooth map  $x_{n+1} = 2x_n \pmod{1}$ .

a) Binary expansions for initial points on 4-cycles include

$$x_0 = 0.0001\ 0001\ \dots$$

$$x_0 = 0.0011\ 0011\ \dots$$

$$x_0 = 0.0111\ 0111\ \dots$$

The point  $x_0 = 0.01010101\dots$  is on a 2-cycle. All other symbol sequences are shifted copies of those above.

The three initial points given above correspond to distinct 4-cycles because each symbol sequence contains blocks of 1s of different length.

The 4-cycles are:

$$x_0 = 0.0001\dots = \frac{1}{16} \left( 1 - \frac{1}{4^{16}} \right) = \frac{1}{15} \xrightarrow{2} \frac{2}{15} \xrightarrow{4} \frac{4}{15} \xrightarrow{8} \frac{8}{15} \xrightarrow{1} \frac{1}{15}$$

$$x_0 = 0.0011\dots = \frac{1}{15} + \frac{2}{15} = \frac{3}{15} \rightarrow \frac{1}{5} \rightarrow \frac{2}{5} \rightarrow \frac{4}{5} \rightarrow \frac{3}{5} \rightarrow \frac{1}{5}$$

$$x_0 = 0.0111\dots = \frac{1}{15} + \frac{2}{15} + \frac{4}{15} = \frac{7}{15} \rightarrow \frac{14}{15} \rightarrow \frac{13}{15} \rightarrow \frac{11}{15} \rightarrow \frac{7}{15}$$

b) Binary expansions for initial points on 5-cycles are generated in a similar way:

$$\begin{aligned} x_0 &= 0.00001\ 00001\dots &= \frac{1}{31} \\ x_0 &= 0.00011\ 00011\dots &= \frac{3}{31} \\ x_0 &= 0.00111\ 00111\dots &= \frac{7}{31} \\ x_0 &= 0.01111\ 01111\dots &= \frac{15}{31} \\ x_0 &= 0.00101\ 00101\dots &= \frac{5}{31} \\ x_0 &= 0.01011\ 01011\dots &= \frac{11}{31} \end{aligned}$$

Any other symbol sequence of least period 5 is a shifted copy of one of those above.

Then

$$x_0 = \frac{1}{31} \rightarrow \frac{2}{31} \rightarrow \frac{4}{31} \rightarrow \frac{8}{31} \rightarrow \frac{16}{31} \rightarrow \frac{1}{31}$$

$$x_0 = \frac{3}{31} \rightarrow \frac{6}{31} \rightarrow \frac{12}{31} \rightarrow \frac{24}{31} \rightarrow \frac{17}{31} \rightarrow \frac{3}{31}$$

$$x_0 = \frac{7}{31} \rightarrow \frac{14}{31} \rightarrow \frac{28}{31} \rightarrow \frac{25}{31} \rightarrow \frac{19}{31} \rightarrow \frac{7}{31}$$

$$x_0 = \frac{15}{31} \rightarrow \frac{30}{31} \rightarrow \frac{29}{31} \rightarrow \frac{27}{31} \rightarrow \frac{23}{31} \rightarrow \frac{15}{31}$$

$$x_0 = \frac{5}{31} \rightarrow \frac{10}{31} \rightarrow \frac{20}{31} \rightarrow \frac{9}{31} \rightarrow \frac{18}{31} \rightarrow \frac{5}{31}$$

$$x_0 = \frac{11}{31} \rightarrow \frac{22}{31} \rightarrow \frac{13}{31} \rightarrow \frac{26}{31} \rightarrow \frac{21}{31} \rightarrow \frac{11}{31}$$