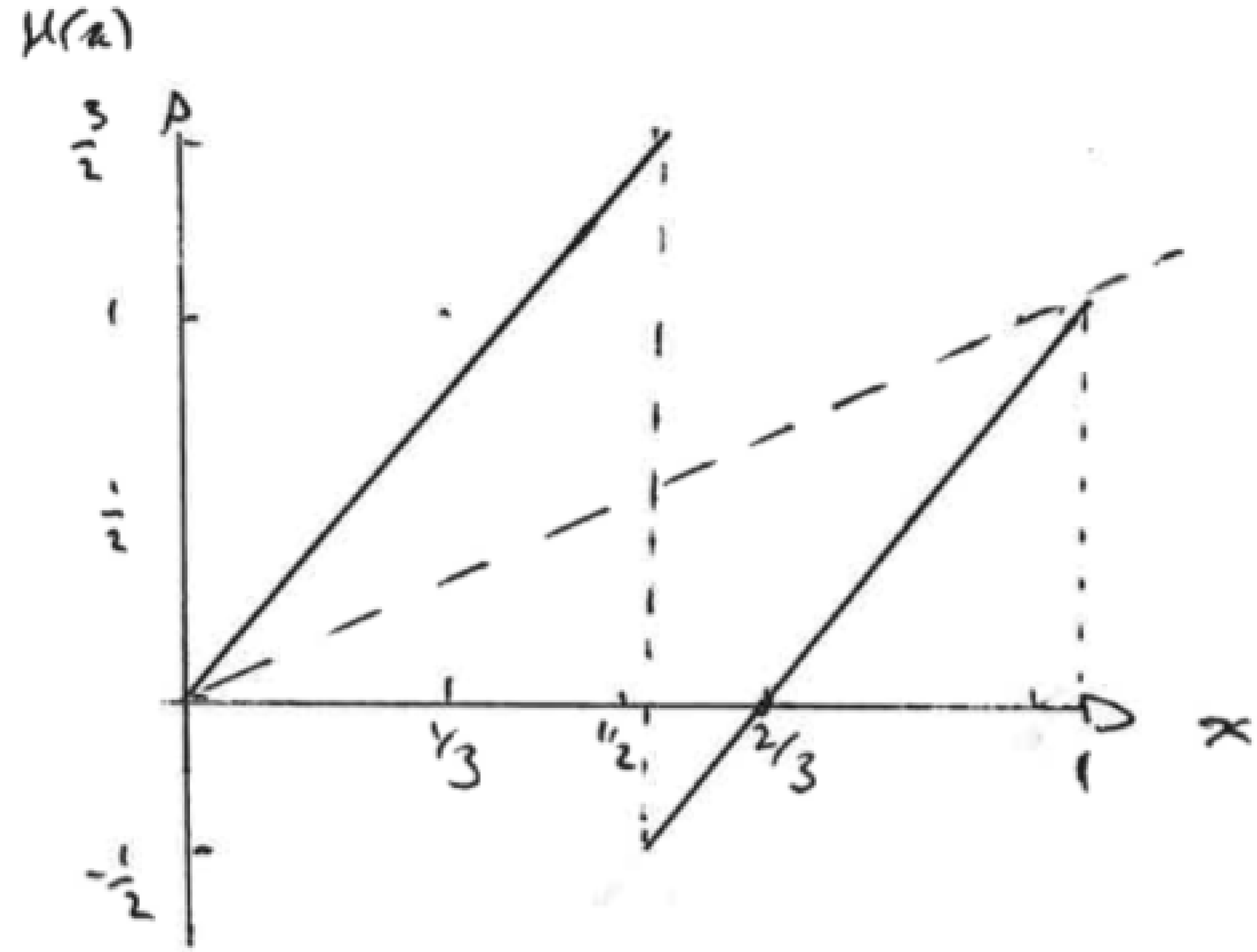
2. a)

$$H(x) = \begin{cases} 3x & 0 \le x \le 1/2 \\ 3x-2 & 1/2 < x \le 1. \end{cases}$$

i)

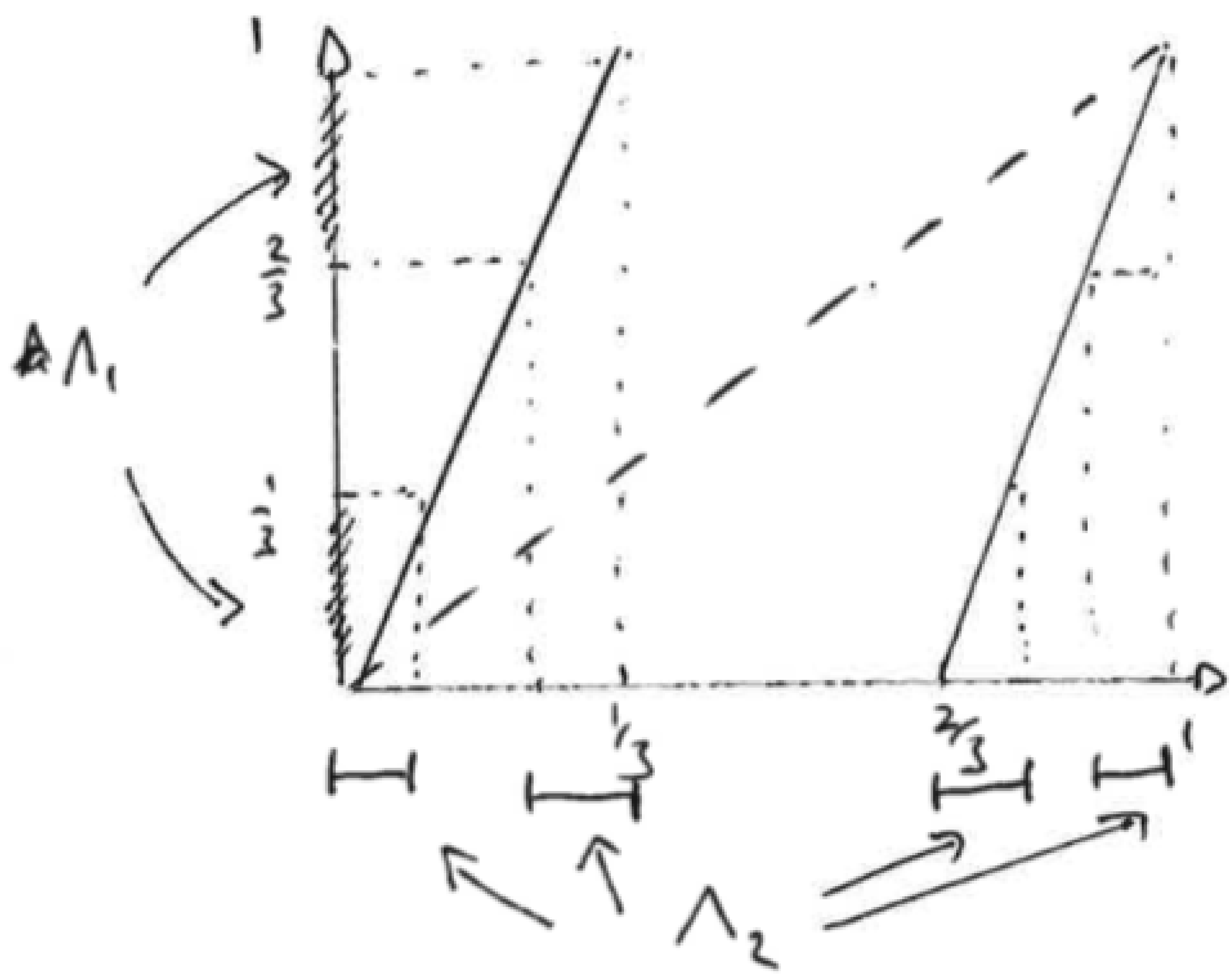


Points that remain in [0,1] for at least one iteration

$$\Lambda_1 = [0, 1/3] \cup [2/3, 1]$$

Points that remain in [0,1] for at least two iterations

$$\Lambda_2 = H^{-1}(\Lambda_1)$$
$$= [0, 1/9] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$



So Λ_2 is a disjoint union of 4 closed intervals.

Inductively, Λ_n is a disjoint union of 2^n closed intervals.

These intervals are nested so that $\Lambda_n \supset \Lambda_{n+1}$.

Hence $\Lambda := \bigcap_{k=1}^{\infty} \Lambda_n$ is non-empty by the Cantor Intersection Theorem.

ii) Let x have the base-3 expansion

$$x = 0.\overset{\frac{1}{3}}{a_0}\,\overset{\frac{1}{9}}{a_1}\,\overset{\frac{1}{27}}{a_2}\ldots$$

The action of H on x is conjugate to the action of the shift

σ on the base-3 expansion, possibly with some adjustment of the leading term resulting from the -2 if $x > 1/2$.

A single iteration of ~~f~~ H maps x outside $[0,1]$ if $1/3 < x < 2/3$.

In base-3 terms, this is if x is between $0.100\ldots$ and $0.111\ldots$

All other base-3 expansions, including $0.100\ldots$ and $0.111\ldots$, map inside $[0,1]$.

Hence any ~~bas~~ point x ~~contain~~ with a base-3 expansion containing a 1 will eventually be shifted under σ to an expansion $0.1a_1a_2a_3\ldots$ and will map outside $[0,1]$ on next iteration.

The points $x = \frac{1}{3} = 0.100\ldots$ and $x = \frac{2}{3} = 0.111\ldots$ correspond to the pre-images of the fixed point $x = 1$.

In summary $\Lambda = \{0.a_0a_1a_2\ldots \mid a_i \neq 1 \ \forall i\}$
$\cup \{0.1000\ldots, 0.111\ldots\}$

iii) $$0.002\,002\,002\ldots = \frac{2}{27}\left(1 + \frac{1}{27} + \left(\frac{1}{27}\right)^2 + \ldots\right)$$
$$= \frac{2}{27}\left(\frac{1}{1-1/27}\right) = \frac{1}{13}.$$

Iterating $H(1/13)$:
$$\frac{1}{13} \to \frac{3}{13} \to \frac{9}{13} \to \frac{1}{13},$$ period 3, as expected.