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Unifying the steady state resonant solutions of the periodically forced KdVB, mKdVB, and eKdVB equations

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ABSTRACT

The periodically forced extended KdVB (eKdVB) equation, which contains both KdVB and modified KdVB (mKdVB) equations as special cases, is known to possess a rich array of resonant steady solutions. We present an analytic methodology based on singular perturbation and asymptotic matching in order to illustrate and approximate these solutions in the limit that the dispersive effects are small relative to the nonlinear and forcing terms. Weak Burgers damping is also included at the same order as dispersion. Solutions across the resonant band may be constructed and show good agreement with solutions of the full equation, showing clearly the role of the various physical effects. In this way, direct comparisons and connections are made between the various classes of KdVB equations, illustrating, in particular, the underlying mathematical connections between the KdVB and mKdVB equations.

1. Introduction

The physical motivation of this work stems from a series of theoretical and experimental studies, beginning with the 1967 paper by Chester [1]. There, a theory was presented to describe the resonant sloshing of water in a tank forced by a periodic wavemaker. In 1986, it was shown in [2] that this problem can be modelled as a periodically forced, damped, Korteweg–de Vries (KdV) equation.

In this paper, we will instead consider the steady, periodic solutions of a more generalized KdV variant: a periodically forced *extended* Korteweg–de Vries equation with Burgers Damping (eKdVB),

$$u_t - \underbrace{\gamma u_{xxx}}_{\text{dispersion}} + \underbrace{\Delta u_x}_{\text{detuning}} + \underbrace{\alpha u u_x + \beta u^2 u_x}_{\text{quad. and cubic nonlinearity}} - \underbrace{\mu u_{xx}}_{\text{damping}} = \underbrace{f'(x)}_{\text{forcing}}. \tag{1}$$

The parameters in (1) are chosen with the above water-wave problem in mind: x is confined within the interval $[0, 2L]$, $f'(x)$ corresponds to an arbitrary forcing with period $2L$, γ and μ provide the dispersive and diffusive effects, respectively, and Δ measures the amount of detuning from resonance. The quadratic and cubic nonlinearities are represented by α and β, respectively, and can be used to further classify the various KdVB-type equations. If $\beta = 0$, (1) reduces to the forced KdVB equation, while if $\alpha = 0$, (1) reduces to a forced *modified* KdVB equation (mKdVB).[1] But the above water-tank system is certainly not the only scenario where equations of the KdVB-type arise; physical scenarios governed by KdVB, mKdVB, or eKdVB equations can be found, for example, in the works of Marchant and Smyth [3], Smyth [4], Wu [5], and Grimshaw

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1 To avoid confusion and to emphasize our classification, we will generally refer to the eKdVB only when *both* α and β are non-zero.

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et al. [6]. However, the types of solutions which arise in these systems are highly dependent on the relative magnitude of the various parameters; certain choices are more amenable to standard analysis, while others are more difficult.

It should also be noted that, in a physical sense, Burgers damping is not the only relevant form of dissipation which may arise. For example in the case of a forced shallow tank, Chester [1] derives, in detail, an expression for the dissipation at the walls through integration across the boundary layer. However, it has also been observed (e.g. see [7]) that, in the weakly dissipative regime, the particular form of damping does not qualitatively impact the nature of the steady solutions, and rather ensures transition from an arbitrary initial state to a unique steady solution. Therefore, the Burgers term is typically employed, as it provides a general model for dissipative effects while being much more amenable to analysis.

In problems where both nonlinear and forcing effects of (1) are strong – such as for the motivating water-wave problem – it is known that a rich array of steady solutions emerge. And, although analytic studies in this regime have been performed, for instance in [7,8], they still remain limited in terms of their scope. Moreover, the interplay between dissipative and dispersive effects has also been studied in various regimes, beginning with the work of Johnson [9]. Amundsen, Cox, and Mortell [10] have developed a general framework based on singular perturbation and asymptotic matching in the context of the forced KdVB. By extending this to the more general eKdVB equation, we can further develop this methodology. Additionally, this provides a more global perspective of the similarities and differences between the various KdV-type equations. Although individual studies of the forced KdVB, mKdVB, and eKdVB exist, there are few that have sought to compare and contrast these equations under a collective heading. Our work in this respect is distinct and revealing.

2. Analytic approximation of steady solutions

We first return to the eKdVB (1) where, in order to ensure boundedness of the long-time solutions, it is assumed that the periodic forcing, $f'(x)$ has zero mean; physically, this often corresponds to a global constraint such as mass conservation. Now integrating (1) over the domain and enforcing periodicity in x for u, we may then assume, without loss of generality, that u satisfies the global condition,

$$\int_0^{2L} u(x,t)\mathrm{d}x = 0. \tag{2}$$

While the question of unsteady solutions is also interesting and there are indeed cases where aperiodic and even chaotic behavior is observed (see e.g. [11]), we restrict our focus to cases where the dissipative effects have rendered solutions steady in the limit $t \to \infty$.

Letting $u_t = 0$, integrating, and rearranging (1) yields the following steady formulation,

$$\gamma u_{xx} + \mu u_x = \frac{\beta}{3}u^3 + \frac{\alpha}{2}u^2 + \Delta u - (f(x) + C), \tag{3}$$

where $f(x)$ is taken as an anti-derivative of $f'(x)$ having zero mean and C is a constant of integration. Integrating once more and using both periodicity of u and the mean condition (2), we can express the constant as, $C = \frac{1}{2L}\int_0^{2L}\left(\alpha\frac{u^2}{2} + \beta\frac{u^3}{3}\right)\mathrm{d}x$.

2.1. Deriving the leading order non-dispersive solutions

In the context of the physical systems discussed in Section 1, we will consider cases where the dissipative effects are small ($\gamma \ll 1$), the diffusion is of the same order, $\mu = O(\gamma) = \nu\gamma$ for some constant $\nu = O(1)$, and all other parameters (including the forcing) are assumed of order unity. We now seek to obtain approximate solutions, asymptotically valid in the limit that γ tends to zero.

A simple perturbation expansion, $u(x) = u_0(x) + \sqrt{\gamma}u_1(x) + \gamma u_2(x) + \cdots$ applied to (3) yields the leading order equation,

$$\frac{\beta}{3}u_0^3 + \frac{\alpha}{2}u_0^2 + \Delta u_0 - (f(x) + C) = 0, \tag{4}$$

which captures the crucial balance between the effects of the nonlinearity and forcing. If $\beta = 0$, the problem reduces to that of a regular KdVB and this special case is reviewed in Section 3. But provided $\beta \neq 0$, the leading order *non-dispersive* solutions u_0 can then be chosen as one of three possible solutions of Eq. (4),

$$u_{0_k} \equiv \frac{1}{2\beta}\left[\rho^{1/3}\mathrm{e}^{\frac{2\pi \mathrm{i}k}{3}} - \left(\frac{4\Delta\beta - \alpha^2}{\rho^{1/3}}\right)\mathrm{e}^{-\frac{2\pi \mathrm{i}k}{3}} - \alpha\right], \quad k = 0, 1, 2 \tag{5}$$

where,

$$\rho = 12\beta^2 G + 6\Delta\alpha\beta - \alpha^3 + 2\sqrt{M}\beta,$$
$$M = 4\beta\left(4\Delta^3 + 9\beta G^2\right) - 3\alpha\left(2G\alpha^2 + \Delta^2\alpha - 12\Delta\beta G\right).$$

and where we have used the substitution $G \equiv f(x) + C$.

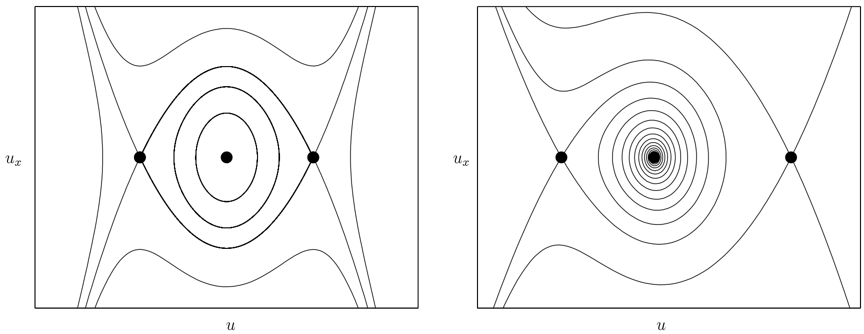

Fig. 1. Phase plane configuration of the undamped (left) and damped (right) mKdVB equation. The non-dispersive solutions (fixed points) will shift in response to the forcing.

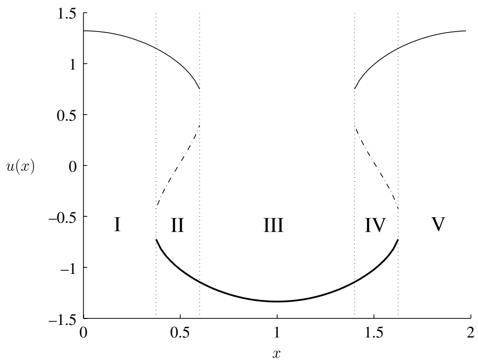

Fig. 2. Partial real non-dispersive solutions for the mKdVB equation with $\beta = 3$, $\Delta = -1$, $f(x) = \cos(\pi x)$, and $C = 0$. The thin, thick, and dashed lines correspond to $k = 0$, 1, and 2, respectively. In Section 2.2, we will derive the dispersive solutions for layers II and IV.

For fixed x, these non-dispersive solutions u_{0_k} are precisely the fixed points of the phase plane of Eq. (3). The forcing can then be visualized as producing a slow, horizontal shift of the fixed points; solutions may begin at such a point, but will slowly vary in response to the forcing, thus producing three non-dispersive solutions. For example, Fig. 1 illustrates the phase plane for the mKdVB ($\beta = 3, \alpha = 0$) in the cases with and without damping.

However, not all such solutions provided by (5) satisfy the zero-mean condition and, in fact, it is not even assured that these solutions will remain real and differentiable. Indeed, there are generally two critical values, $\Delta = \Delta_{1,2}$ for which the non-dispersive solutions first fail as leading order approximations of (3). When $\beta = 0$ (KdVB), these critical values lie at $\Delta_{1,2} = \pm\sqrt{-2\alpha G_{\text{max/min}}}$, where $G_{\text{max/min}}$ denotes either the global maximum or minimum of G, chosen so that the argument of the square root is positive. This corresponds to the point where the non-dispersive solutions cease to be entirely real. When $\beta \neq 0$, the two critical values are $\Delta = \frac{\alpha^2}{4\beta}$ and $\Delta = \Delta_M$ where $M(\Delta_M) = 0$, corresponding to when the quadratic roots of (4) transition from being complex to real and vice versa. The interval delineated by these two critical values forms the *resonant band*.

Thus, while outside of the resonant band, there is always one non-dispersive solution which is entirely real and satisfies the mean condition, for other intermediary values of Δ, a regular perturbation approach fails and dispersive effects must be taken into account. Fig. 2 provides a striking example of this scenario, where the layers for which the non-dispersive solutions fail are clearly delineated. Note in this, and all subsequent examples, the forcing $f(x) = \cos(\pi x)$ with associated domain half-width $L = 1$.

2.2. Deriving the leading order dispersive solutions

For solutions within the resonant band, we turn our focus upon a layer at $x = x_i$ where dispersive effects cannot be neglected at leading order. Within this layer, a new stretched coordinate is introduced, $X = \frac{\omega_1(x)}{\sqrt{\gamma}} + \omega_2(x) + \sqrt{\gamma}\omega_3(x) + \cdots$ so that the solutions $u = u(x, X)$ are now assumed to vary on both a slow (x) and fast (X) coordinate scale. Applying a perturbative approach $u(x, X) = u_0(x, X) + \sqrt{\gamma}u_1(x, X) + \gamma u_2(x, X) + \cdots$, it can be seen that upon integration, (3) yields to leading order,

$$\omega_1'^2 u_{0X}^2 = \frac{\alpha}{3}u_0^3 + \frac{\beta}{6}u_0^4 + \Delta u_0^2 - 2(f(x) + C)u_0 + E, \tag{6}$$

where $E = E(x)$ is a further constant of integration. By appealing now to the bounded and zero-mean nature of the solutions, it can be further assumed that the four roots of (6) are real, allowing us to write the solution in an explicit form (see [12]),

$$u_0(x, X) = \frac{-(b-d)c + \mathrm{sn}^2(B(X-\delta), m)(b-c)d}{-(b-d) + \mathrm{sn}^2(B(X-\delta), m)(b-c)}, \tag{7}$$
$$B = \frac{\sqrt{(a-c)(b-d)}}{2\omega_1'}, \qquad m = \frac{(b-c)(a-d)}{(a-c)(b-d)},$$

where $a \le b \le c \le d$ are the roots of the quartic polynomial in (6). It is important to notice that these quartic roots, as well as the other parameters of the solution, δ and ω_1 will, in general, slowly vary in response to the forcing.

In order to account for the nature of this slow variation, we investigate the secularities arising at subsequent orders. First, as noted in [13], the periodicity of the solutions must be constant with respect to the slow variation, otherwise secular growth occurs. Thus by the periodicity of elliptic functions, $T = \frac{2K(m)}{B} = \frac{2K(m)}{\sqrt{(a-c)(b-d)}}\omega_1'$, where T is an arbitrary constant associated with the period in X, and $K(m)$ is the complete elliptic integral of the first kind. Choosing, for convenience, $T = 2$, we obtain the expression for the leading order variation of the dispersive layer coordinate,

$$\omega_1'(x) = \frac{\sqrt{(a-c)(b-d)}}{K(m)}. \tag{8}$$

Next, to determine the modulus, $m(x)$ in terms of the spatial variation, we consider the equation which arises at $O(\sqrt{\gamma})$,

$$-\omega_1^2 u_{1XX} + \Delta u_1 + \alpha u_0 u_1 + \beta u_0^2 u_1 = 2\omega_1'\omega_2' u_{0XX} + 2\omega_1' u_{0xX} + \omega_1'' u_{0X} + \nu\omega_1' u_{0X}. \tag{9}$$

Noting that the solution of the homogeneous self-adjoint problem is u_{0X}, we take the standard functional inner product of (9) and apply periodicity considerations to obtain the solvability condition,

$$\frac{\mathrm{d}}{\mathrm{d}x}\omega_1' \int_0^T u_{0X}^2 \mathrm{d}X - \nu\omega_1' \int_0^T u_{0X}^2 \mathrm{d}X = 0. \tag{10}$$

This equation is easily integrated, yielding

$$\omega_1' \int_0^T u_{0X}^2 \mathrm{d}X = \kappa \mathrm{e}^{-\frac{\mu}{\gamma}x}, \tag{11}$$

where κ is a constant of integration. This serves two purposes: It provides a location for the layer, and it relates the modulus to the spatial variation x provided that a value of κ is determined.

In a physical sense, this equation corresponds to the slow variation in energy induced by the dissipation and may also, for instance, be derived via a modulation theory approach based on the underlying conservation laws. For the general theory, see [14] and, for application in the context of the weakly damped KdVB, see [15]. Our approach, based on a stretched coordinate and underlying asymptotic expansion, provides the same result but with a more precise notion of the relative scales involved in the problem and, by retaining the fast variation, allows for qualitative understanding of the dispersive layers via the phase plane. Moreover, as will be seen in the following section, it also facilitates matching with the non-dispersive layers.

2.3. *Matching of solutions*

At this point, we have managed to derive the explicit formulation of the rapidly varying dispersive solution, but several parameters have yet to be determined: κ, the constant of the solvability condition (11); δ, the phase shift; and $m(x)$, the modulus. These parameters will be determined by matching the inner and outer solutions across the boundary layer interface.

2.3.1. *Determining κ by enforcing a matching*

Recall our discussion of the phase plots of Fig. 1. The non-dispersive solutions of (5) correspond to fixed points which move slowly in response to the variation of the forcing. Consequently, solutions within the dispersive layer correspond to trajectories beginning and ending at a fixed point, configured in such a way as to satisfy the mean condition (2).

Thus, in order for a successful match to occur, we must require the solution trajectory to smoothly enter the separatrix leading into the appropriate non-dispersive solution at the point of matching. Formally, this produces the requirement that the modulus of the elliptic functions in (7) tends to unity $m(x) \to 1$ over the dispersive-to-non-dispersive interface.

Now determining the value of κ in the solvability condition (11) is easily done: Given the locations of the matching $x_{i_L} < x_i < x_{i_R}$, we calculate the value of $\omega_1'(x)$ and the four roots a, b, c, d using the condition $m(x) \to 1$ at $x = x_{i_L}, x_{i_R}$. This completely determines our dispersive solution u_0 and now κ is given by the solvability condition (11) evaluated at the point of matching. Once a value of κ has been determined, both $\omega_1'(x)$ and $m(x)$ are easily solved as functions of x: the ω_1 using (8), and the modulus $m(x)$ using (11).

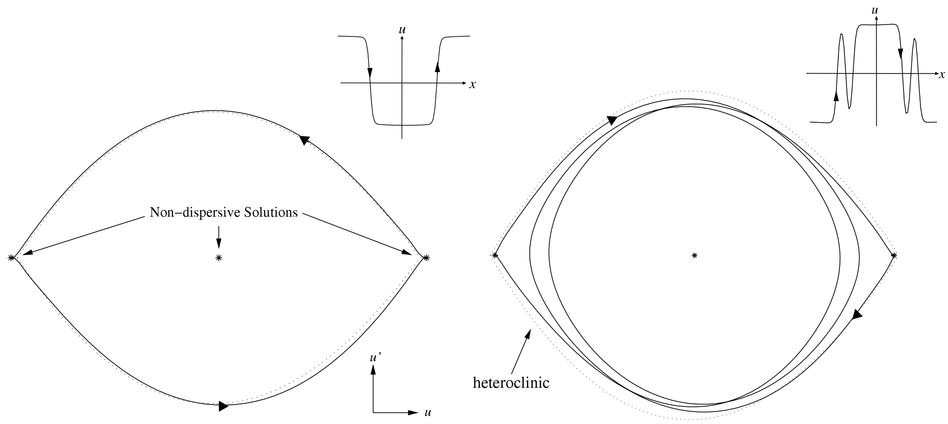

Fig. 3. Single (left) and multiple (right) orbit solutions of the mKdVB.

2.3.2. *Determining the phase shift* δ

The phase shift, in conjunction with the location of the boundary layer $x = x_i$, and arising in the definition of the scaled parameter, X, must be chosen correctly in order to match the phase of the dispersive solution to the non-dispersive solution in the limit as the modulus approaches unity. Note, then, that for a successful match, we will also require an appropriate choice of δ such that only an integer number of peaks exist within the dispersive layer. In general, however, the algebraic complexity of the solutions (7) necessitates numerically solving for the phase δ.

3. Special cases of the eKdVB

Before turning to the general extended KdVB, we first study the two extremal cases where only a single nonlinearity is present. A central theme throughout this work is the notion that the solutions of the forced eKdVB can be qualitatively and broadly classified according to its degree of KdVB or mKdVB-like behavior. By understanding these extremal cases, we gain much insight into the underlying structure of the more general eKdVB.

3.1. *The modified KdVB equation* ($\alpha = 0$)

In the case $\alpha = 0$, the steady-state eKdVB (3) reduces to the mKdVB,

$$-\gamma u_{xx} + \frac{\beta}{3}u^3 + \Delta u - \mu u_x = f(x) + C, \tag{12}$$

which is simply one of the many variants of the well known Duffing equation. The mKdVB possesses two saddles in the phase plane flanking a lone nodal point. Three non-dispersive solutions are provided by (5) and they are invalid within the resonant band, where all three solutions fail to be real in various subintervals of $[0, 2L]$. In these cases, the dispersive solutions are provided by a complex expression of elliptic functions (7) which involves transitions between *both* saddles.

The simplest transition is one in which only a *single* period is traversed before returning to the non-dispersive solution. In this case, the solution begins at a saddle point, follows a trajectory close to the heteroclinic orbit, and matches with the flanking saddle point. In fact, to leading order $\gamma \to 0$, and $m(x) \to 1$, the dispersive solutions centered at $x = \delta$ are approximated by the heteroclinic itself,

$$u_{\text{heteroclinic}}(x) = \pm\sqrt{-\frac{3\Delta}{\beta}}\tanh\left(\sqrt{-\frac{\Delta}{2\gamma}}(x-\delta)\right), \tag{13}$$

where the square root arguments are positive for parameters within the resonant band and the choice of sign $\pm$ corresponding to the upper and lower trajectories of the heteroclinic.

However, if the boundary layer is sufficiently large, multiple orbits are then possible, and the slow variation in the forcing forms a crucial element behind our multiple scales analysis in Section 2. In this case, transitions will consist of a solution, leaving a saddle point along the separatrix, orbiting the nodal point a specified number of times, and returning smoothly to the flanking saddle.

Fig. 3 depicts typical single (left) and multiple (right) orbit cases for the mKdVB equation near resonance. In the case of the left figure, only a single jump occurs between non-dispersive media and the trajectory follows closely with the heteroclinic orbit. In contrast, the right figure depicts a trajectory that orbits the nodal solution one and a half times before finally matching with the flanking saddle.

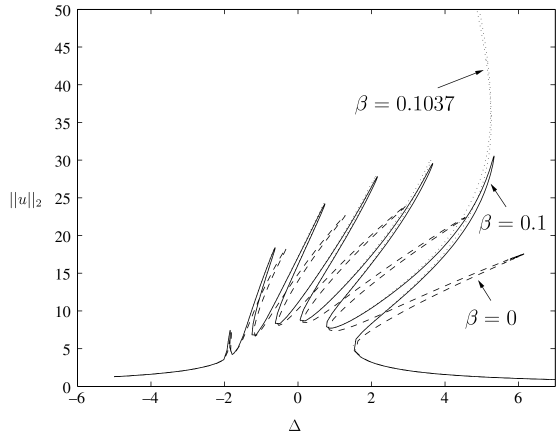

Fig. 4. Resonant response for the eKdVB equation with $\gamma = .005$, $\mu = 0.0015$, $f(x) = \cos(\pi x)$, $\alpha = 2$, and $\beta = 0, 0.1, 0.1037$.

3.2. *The KdVB equation* ($\beta = 0$)

In the case $\beta = 0$, the eKdVB (3) reduces to the well-known KdVB,

$$-\gamma u_{xxx} + \alpha u u_x + \Delta u_x - \mu u_{xx} = f'(x). \tag{14}$$

A detailed asymptotic analysis of the forced KdVB near resonance is found in [10], and indeed our methodology and development in the previous section extends and compliments their work to the more general eKdVB. For the KdVB, however, the solutions simplify considerably due to the fact that now there are only two non-dispersive solutions: one corresponding to a saddle, the other to a stable node. The leading order dispersive solutions found by solving (5) may also be expressed in terms of Jacobi Elliptic functions but, here, the limiting case of a single period now corresponds to a *homoclinic* orbit (in the limiting form of sech^2 functions) about the nodal solution. More generally, dispersive solutions correspond to multiple orbits about the nodal solution, beginning and ending at the saddle point.

4. The general extended KdVB (α, $\beta \neq 0$)

When both quadratic and cubic nonlinearities are present, a much richer array of solutions arise, the nature of which depends on the relative strengths of each parameter. The principle result of our work concerns the connection between the three KdV-type equations and, in particular, the results of examining the effects as the nonlinearity transitions from purely quadratic (KdVB) to purely cubic (mKdVB), and the combined regime in between (eKdVB).

4.1. *A transformation from eKdVB to mKdVB*

It is well known that the linear transformation, $v = u + \frac{3\alpha}{4\beta}$ can be applied to recast the eKdVB (3) into an mKdVB equation,

$$-\gamma u_{xx} + \frac{\beta}{3}u^3 + \left(\Delta - \frac{3\alpha^2}{16\beta}\right)u - \mu u_x = f(x) + C,$$

where the mean condition (2) changes to, $\int_0^{2L} v(x)\mathrm{d}x = \frac{3\alpha}{2\beta}L$. Although analytic solutions of the eKdVB can be treated with the mKdVB development of the previous section, the transformation does not lend itself to a significant grasp of the qualitative structure of the solutions. Moreover, the change in the mean condition also changes the class of allowable solutions, and so we will primarily concern ourselves with the qualitative aspects of the solutions.

4.2. *Qualitative structure of the eKdVB*

General eKdVB behavior can be classified broadly into various subcategories, depending on the relative magnitude of the nonlinearities, α and β. For example, the regime that occurs in the transition from eKdVB to mKdVB ($\alpha \ll 1$, $\beta \sim O(1)$) is unremarkable — the effect of the (nearly negligible) quadratic nonlinearity is only a small asymmetric repositioning of the fixed points in the phase plane.

However, the same cannot be said of the transition from KdVB to eKdVB ($\alpha \sim O(1)$, $\beta \ll 1$). The qualitative nature of these solutions can best be understood by examining the resonance curves as the α and β parameters are varied, such as in Fig. 4. Here, the 2-norm of the solutions, $\|u\|_2$ is shown as the detuning, Δ varies. Notice the two distinct regions in each resonant response: the edges $|\Delta| \gg 0$ where the non-dispersive solutions are valid, and the resonant band near $\Delta = 0$

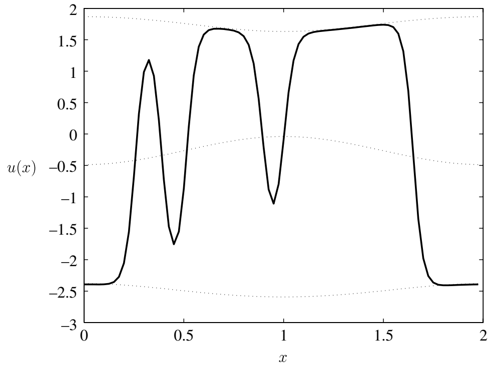

Fig. 5. An eKdVB solution for $\gamma = 0.0005$, $\mu = 0.0015$, $f(x) = \cos(\pi x)$, $\alpha = 2$, $\beta = 3$, and $\Delta = -4.19$ depicting combination effects.

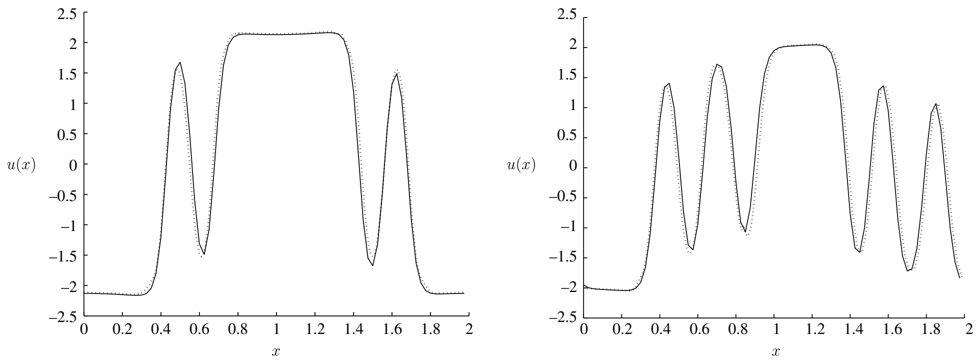

Fig. 6. Approximate analytic (dotted) and numerical (solid) solutions for the mKdVB with $\gamma = 0.005$, $\beta = 3$, $\mu = 0.0015$, $f(x) = \cos(\pi x)$ and the two cases $\Delta = -5$ (left) and $\Delta = -4.57$ (right). The solutions show excellent agreement.

where the dispersive regime is in effect. Each 'finger' in the graph indicates a qualitatively different response, corresponding to solutions with different trajectories in the phase plane.

As the cubic nonlinearity increases, the fingers bend backwards, but the qualitative nature of the solutions are still KdV-like, and the solutions unaffected by the third, distant, saddle. Then, at some critical value of β, the resonant response changes drastically, marking the first point that solutions begin interacting with the third (mKdVB) saddle. This critical value corresponds to the emergence of a heteroclinic orbit and, as such, is associated with the smallest β such that the second saddle intersects the homoclinic. Using (5), with C and Δ taken to be the values associated with the point at which the homoclinic is at its maximum extent and the parameter values as noted, this critical value is approximately $\beta_c \approx 0.1026$, in agreement with what is observed.

Finally, in the 'fully' eKdVB regime, where both cubic and quadratic nonlinearities are dominant, we see solutions with a combination of effects. Solutions of the eKdVB can be seen to exhibit behavior either of the KdVB-type, where boundary layer transitions occur from and to a single saddle point, or of the mKdVB-type, where solutions transition *between* distinct saddles. Fig. 5 provides an example of an eKdVB solution which possesses two layers with different mKdVB-like transitions, as well as a third, homoclinic KdV-like layer in the centre.

We emphasize that the leading order asymptotic behavior of these solutions can be produced through the theory of Section 2.2 (see [16]). Moreover, the location of the bifurcation seen in Fig. 4 corresponds to the case where the cubic nonlinearity is sufficiently large that a heteroclinic solution between the two saddles is permitted.

5. Discussion

By applying the methodology developed in the previous sections, we can see that the agreement between numerical and approximate solutions is excellent for a wide range of parameters. Fig. 6 provides an example of the high accuracy of approximating the solution using leading order approximations. In particular, we have generated approximations to solutions of (3) for $\gamma = 0.005$, $\alpha = 0$, $\beta = 3$, $\mu = 0.0015$, $f(x) = \cos(\pi x)$, and the two cases $\Delta = -5$ (left) and $\Delta = -4.57$ (right).

However, there are still limitations. While the outlined methodology is successful in developing asymptotic approximations for most solutions, it is implicit in the procedure that dispersive layers are independent and can be studied individually. But this *local* and *layer-based* approach will fail in cases where dispersive layers first separate or coalesce. In such cases, a more global approach may be required.

Nevertheless, we emphasize the effectiveness and flexibility of the techniques developed here in providing a framework for constructing steady resonant solutions of KdV-type equations, as well as providing much insight into the many subtle connections and distinctions between the forced KdVB, mKdVB and the eKdVB equations.

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