Seeing the unseen

Using exponential asymptotics to reveal hidden water waves

AT FIRST, NOBODY SEEMED TO NOTICE. Consider a two-dimensional ideal fluid flowing past or over an obstruction in the stream. Previous investigators had represented the free-surface as an asymptotic expansion,

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots \qquad \begin{aligned} \epsilon &= \text{Froude}^2 \\ &= U^2/gL \end{aligned} \qquad (1)$$

supposedly valid in the limit the Froude number, or alternatively, the speed of the stream tends to zero. But it wasn't until 1968 when naval architect T.F. Ogilvie pointed out the peculiarities of these solutions: First, the approximations predict a *waveless* free-surface. But even if the speed of the stream is small, one would still expect waves to form downstream – where are the waves? Second, the asymptotic expansion in (1) seems to 're-order' as the Froude number tends to zero, with lower and lower speeds requiring more and more terms to achieve desired accuracy. But the approximation should be getting better as limit is approached – not *worse!* Today, however, Ogilvie's observations are easily identified as *necessary consequences* of representing well-defined phenomena using ill-defined and *divergent* asymptotic expansions.

Come, let us seek these dastardly waves.

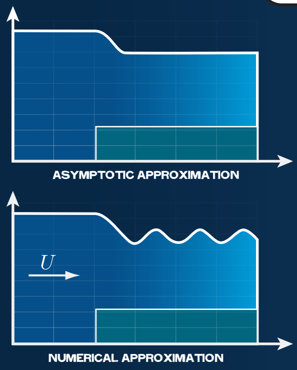
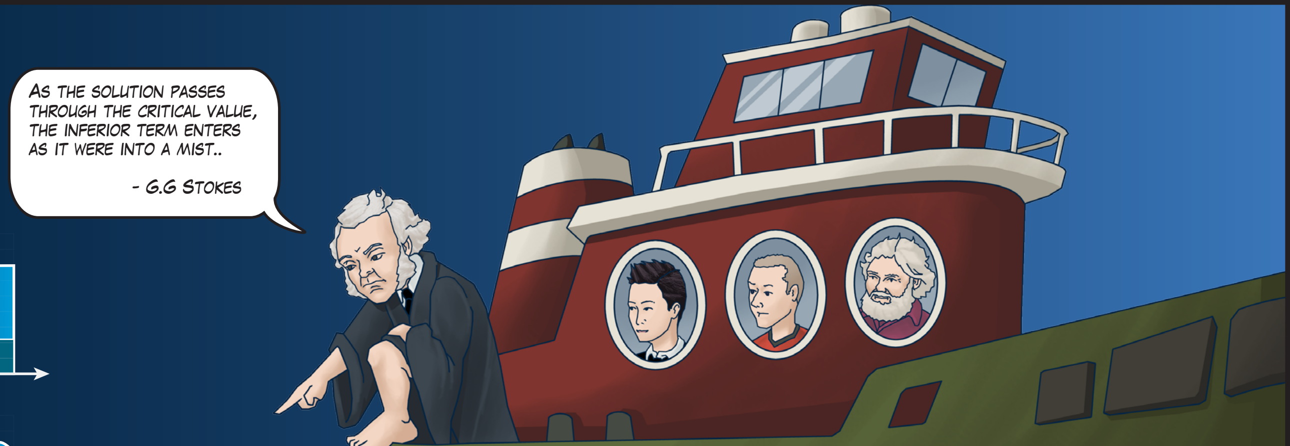


FIGURE 1: FREE-SURFACE FLOW OVER A STEP. WHY ARE THERE NO WAVES IN THE ASYMPTOTIC APPROXIMATION?



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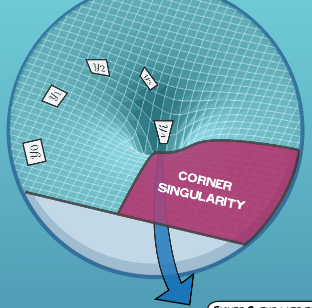

FIGURE 2: THE LATE TERMS ARE ENTIRELY DOMINATED BY THE FACTORIAL OVER POWER BEHAVIOUR AT THE CORNER SINGULARITY.

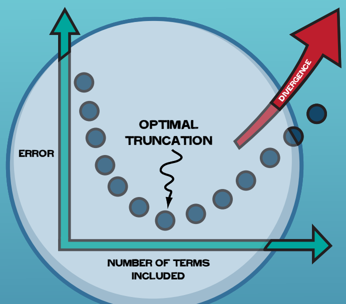

FIGURE 3: AS MORE TERMS ARE INCLUDED IN THE ASYMPTOTIC EXPANSION, THE ERROR DECREASES TO THE OPTIMAL TRUNCATION POINT (WHERE IT IS EXPONENTIALLY SMALL), THEN DIVERGES.

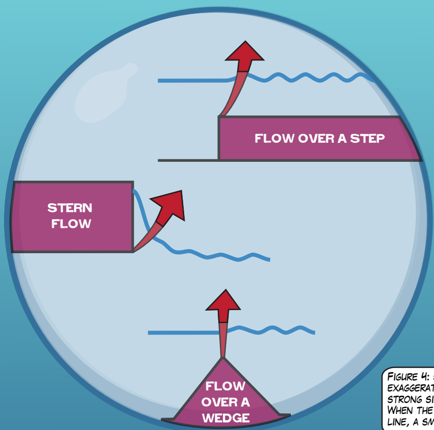

FIGURE 4: STOKES LINES (HERE EXAGGERATED IN SIZE) ORIGINATE FROM STRONG SINGULARITIES IN THE FLOW FIELD. WHEN THE SOLUTION CROSSES THE STOKES LINE, A SMALL EXPONENTIAL SWITCHES ON.

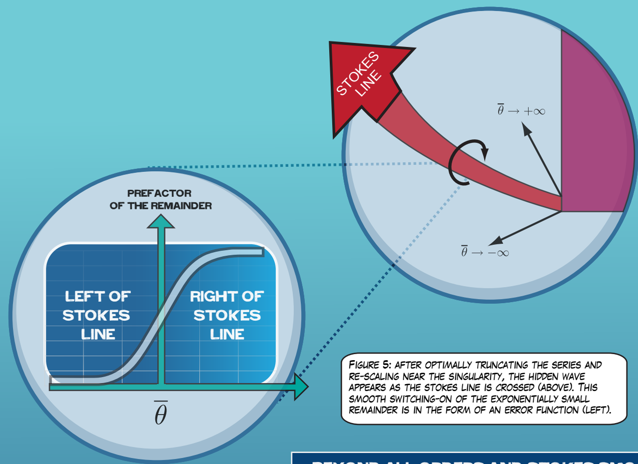

FIGURE 5: AFTER OPTIMALLY TRUNCATING THE SERIES AND RE-SCALING NEAR THE SINGULARITY, THE HIDDEN WAVE APPEARS AS THE STOKES LINE IS CROSSED (ABOVE). THIS SMOOTH SWITCHING-ON OF THE EXPONENTIALLY SMALL REMAINDER IS IN THE FORM OF AN ERROR FUNCTION (LEFT).

A PROBLEM OF SINGULARITIES

Examine the ship to the upper-right, where our four intrepid heros are perched. The key observation is that at low speeds, the asymptotic approximation of the free-surface contains a *singularity* at the corner of the stern. This use of ill-defined approximations in order to represent perfectly well-defined phenomena is one of the caveats of asymptotics, but one would hope that a singularity far from the region of interest (the free surface) has little effect on the approximation.

$$y(\phi) = \epsilon \underbrace{\left(\frac{\phi}{\phi+1}\right)} + \text{higher order terms...}$$

$$\text{velocity potential} \begin{cases} \phi \geq 0 & \text{is the free surface} \\ \phi = -1 & \text{is the corner} \end{cases}$$

This is *not* the case. Because the calculation of each additional term in the series depends on the derivatives of the previous terms, the situation gets a whole lot worse. In fact, in the limit that n tends to infinity, the asymptotic expansion will diverge (Figure 2).

ASYMPTOTIC DIVERGENCE

However, the divergence of the asymptotic series obeys a beautiful universality. In the 1950s, R.B. Dingle showed that the late terms of nearly all divergent series are of the form factorial/power. Thus, we see in (2) that the unsettling growth of the factorial is expounded by the fact that the power of the singularity grows at each subsequent order. If we plot the error in the approximation as a function of the number of terms taken (Figure 3), the error decreases to a minimum (at the *optimal truncation point*), then grows to infinity. These facts about asymptotic divergence apply to a wide class of problems.

$$y_n \sim \frac{Q\Gamma(n+\gamma)}{\chi^{n+\gamma}} \quad \text{as } n \to \infty \qquad (2)$$

THE STOKES PHENOMENON

Ultimately, what is the price we pay for divergence? In the late 19th century, the mathematical physicist G.G. Stokes studied an approximation which exhibited waves in one region of space, but none in the other. He described the waves as seeming to "emerge from a mist". This "mist" Stokes refers to is a mathematical aberration caused by the underlying divergence of the asymptotic series. It can be shown that there exists curves, or *Stokes lines*, which originate from the singularities of the flow field and arc towards the free-surface. Across these lines, a small exponential term switches on. Thus, by plotting the Stokes lines and the numerical solutions for various flows (Figure 4), we can see that waves appear approximately where the Stokes line intersects the free-surface.

$\chi = 0$ at singularities

χ real and positive along Stokes lines

BEYOND ALL ORDERS AND STOKES SMOOTHING

In order to retrieve the small waves that are flicked on as the Stokes line is crossed, we proceed as follows: We truncate the asymptotic expansion at the optimal truncation point (3); here the error in the approximation is exponentially small – the very *best* we can do with traditional asymptotics. We then re-scale near the singularity (Figure 5) and study the jump in the remainder as the Stokes line is crossed. The surface waves are then given by a formula that relates the small waves with the late-order terms of the asymptotic expansion (4).

$$y = \sum_{n=0}^{N} \epsilon^n y_n + R_N \qquad (3)$$

$$\text{waves} \sim 4\pi \Re \left(Q e^{-\chi/\epsilon} \right) \qquad (4)$$

Thus, techniques in exponential asymptotics allow us to extract information from the divergent tails of the asymptotic expansion. The theory is elegant and explains the nature of the hidden waves by recourse to key singularities in the flow field. Moreover, comparisons with numerical simulations have yielded excellent agreement.

REFERENCES

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